4/3/2013

Matrix factorizations - focus period seminar

D. Murfet Matrix factorizations + TFT (= top. field theories)

(1) What is a Landau-Ginzburg model?

closed / open

$W \nearrow$ closed, open $\nwarrow$ M.F.

(2) TFT + algebra

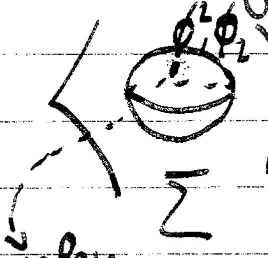
In a L.G. model on a Riemann surface Σ the fields are: $\phi : \Sigma \longrightarrow \mathbb{C}^n$ $\phi_1, \dots, \phi_n$

Dynamics are controlled by Lagrangian, which depends on $W \in \mathbb{C}[x_1, \dots, x_n]$ (and its derivatives!) $W =$ "the potential"

$$\mathcal{L}(\phi, \partial\phi, \dots) = \dots \partial_i W \dots$$

__Observables__ include polynomials in ϕ_i.

__e.g.:__



$$\left\langle \phi_1^2 \phi_2 \right\rangle = \int \mathcal{D}[\phi]\, e^{-S} = \mathrm{Res}_0\left(\frac{dx_1 \dots dx_n}{\partial_{x_1}W \dots \partial_{x_n}W}\right) = /.$$

$$\boxed{S = \int_\Sigma d^2_\sigma \mathcal{L}}$$

get an S for each choice of ϕ

coords on sphere

can be computed by the rule:

$$\mathrm{Res}\left(\frac{f(x)\,dx_1 \dots dx_n}{x_1^{a_1} \dots x_n^{a_n}}\right) = \text{coeff of } (*)$$

$(*)$ $x_1^{a_1 - 1} \dots x_n^{a_n - 1}$ in f

$$\%_{\text{Vafa '90}} = \mathrm{Res}_0\left(\frac{x_1^2 x_2\, dx_1 \cdots dx_n}{\partial_{x_1} W \cdots \partial_{x_n} W}\right).$$

closed CG $\quad \mathcal{H} = \mathbb{C}[[x_1, \dots, x_n]]/(\partial_{x_1} W, \dots, \partial_{x_n} W)$

$\qquad\qquad \mathrm{Res} : \mathcal{H} \to \mathbb{C}$.

Def - A MF of W is a block matrix $D = \begin{pmatrix} 0 & F \\ G & 0 \end{pmatrix}$ $\quad D^2 = W \cdot I$,
i.e. a $\mathbb{Z}_2$-graded free $\mathbb{C}[x]$-module X and deg 1 map $D : X \to X$ such that $D^2 = W \cdot 1_X$

- A connection $\nabla : X \to X \otimes_{\mathbb{C}[x]} \Omega^1_{\mathbb{C}[x]/\mathbb{C}}$ is $\mathbb{C}$-linear s.t.

$$\nabla(hx) = h\nabla(x) + x \otimes dh$$

(if $X = \oplus R e_i$, $\nabla(h e_i) = \sum_j e_j \otimes \partial_j h$)

denote $\Omega^* = \Lambda \Omega^1$

∇ extends to $\nabla \circlearrowleft X \otimes \Omega^*$ with $|\nabla| = 1$ on Ω satisfying for $\omega \in \Omega^*$, $x \in X$:

$$\nabla(x\omega) = \nabla(x)\omega + (-1)^{|x|} x\, d\omega. \quad (*)$$

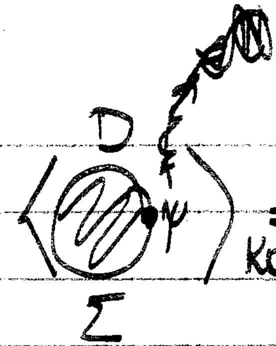
- A superconnection is $\nabla_s \circlearrowleft X \otimes \Omega^*$ of $\mathbb{Z}_2$-degree 1 (using grading of X, Ω^*) satisfying the above Leibniz rule $(*)$.

Example ∇, $\underbrace{\nabla + D}_{\nabla_s} \circlearrowleft X \otimes \Omega^*$

$$\underset{\text{curvature}}{\nabla_s^2} = \nabla^2 + \underbrace{[\nabla, D]}_{\substack{\text{Atiyah class} \\ (=\text{important part})}} + \underset{W}{D^2}$$

- With D as boundary condition:
with $\Sigma =$ 

$$\left\langle \; \right\rangle =_{\text{Kapustin-Li}} \operatorname{Res}_0 \left(\frac{\operatorname{str}(\psi\, \partial_{x_1}(D) \cdots \partial_{x_n}(D))\, dx_1 \cdots dx_n}{\partial_{x_1} W \; \cdots \; \partial_{x_n} W} \right) =$$



ψ a matrix s.t. $\psi D = D\psi$.
supertrace $=: \operatorname{str}$; $\operatorname{str}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \operatorname{tr} A - \operatorname{tr} B$.

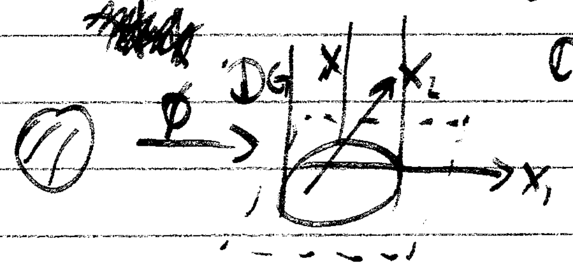
$\mathcal{U} = \operatorname{str}(\underset{\text{Holonomy}}{\mathrm{Hol}_\phi(\partial\Sigma)}\psi) \;\hat{=}\; \operatorname{str}(\exp([\nabla, D])\psi)$

$= \int \mathcal{D}[\phi]\, e^{-S}\, \mathcal{U}$

[X = vect. bdl]

$\mathbb{C}^2$

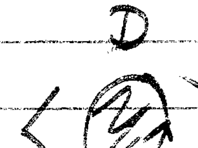
$\underline{n = 2}$:



__Example__ $W = x_1^2 + x_2^2$, $R = \mathbb{C}[x_1, x_2]$
$Y = Re_0 + Re_1$

$$D = \begin{pmatrix} 0 & x_1 - ix_2 \\ x_1 + ix_2 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_1} x_1 + \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_2} x_2$$

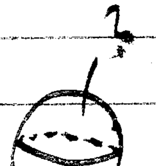
$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0$$
$$\sigma_i^2 = 1$$



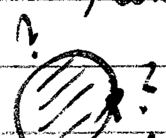
$$\left\langle \; \right\rangle = \operatorname{Res}_0 \left(\frac{\operatorname{str}(\sigma_1, \sigma_2)\, dx_1 dx_2}{2x_1\, 2x_2} \right)$$

$$= \frac{1}{4} \operatorname{str}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}_{\text{const}} = \frac{i}{2}.$$

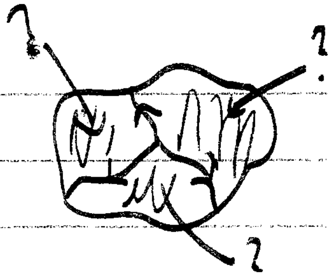
[every ψ, D]



Frobenius algebra $\frac{\mathbb{C}[x_1, \dots, x_n]}{\partial_{x_1} W, \dots, \partial_{x_n} W}$



Calabi-Yau category



very $\mathcal{V}, D, W$.

Bicategory

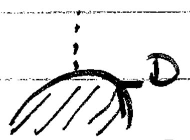
String diagrams $R^e = R \otimes_{\mathbb{C}} R$

$$\Delta = {}_{2}\Lambda_{R^e}\left(R^e \theta_1 \oplus R^e \theta_2\right) \qquad |\theta_i| = 1$$

$$d_\Delta = \sum_{i=1}^{2} (x_i - x_i')\,\theta_i^* + \sum_{i=1}^{2} (x_i + x_i')\,\theta_i.$$

$$d_\Delta^2 = W \otimes 1 - 1 \otimes W$$

Interprete pictures:  means D  means $D^\vee$

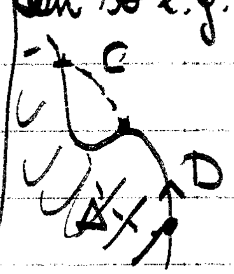
 means  after 

$$\mathbb{C}$$
$$\uparrow ev$$

 means $D^\vee \otimes_R D = \mathrm{Hom}(D, D)$, i.e. $X^\vee \otimes_R X = \mathrm{Hom}(X, X)$

 means $\mathbb{C} \xrightarrow{coev} D^\vee \otimes_R D$ i.e. $X^\vee \otimes_R X$
$$\uparrow R$$
$$\mathbb{C}$$

Can so e.g. compute:



$$X \otimes \cancel{\mathbb{C}} \cong X$$
$$\uparrow$$
$$X \otimes X^\vee \otimes X$$
$$\uparrow$$
$$\Delta \otimes X$$
$$\uparrow$$
$$X$$

$$ev(i \otimes \eta) = \frac{i}{4} (-1)^{|\eta|} i(\eta) \text{ const}$$
$$coev(1) = \sum e_i^* \otimes e_i (-1)^{|e_i|} \qquad ev \circ coev(1) = \frac{1}{2}.$$