MF Seminar

5/3/13

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Knörrer periodicity + finite MF type

References: Knörrer, Buchweitz-Greuel-Schreyer (Invent. 1987) [CM modules on hypersurface sing's I, II].

Setup: $(S, \mathfrak{m}, k)$ complete RLR, $\dim = d+1$, $f \in \mathfrak{m}^2$, $f \neq 0$
$R = S/(f)$ hypersurface rings of dim d.

Recall: A matrix factorization (MF) of f is a pair of maps $G \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F$ between free S-modules F, G so that $\psi\varphi = f 1_G$ and $\varphi\psi = f \cdot 1_F$

Def: The double branched cover of R is
$$R^\# = S[[z]]/(f + z^2)$$

Rmk: If $S = \mathbb{R}[x] \rightsquigarrow \underset{x^2+z^2}{\mathbb{R}[x]/(x^2)} = \underset{-x^2+z^2}{\mathbb{R}[x]/(-x^2)}$

So assume: $k = \bar{k}$ and char $k \neq 2$!

We have a surjection $R^\# \to R$ killing (the image of) z.
Also, $R^\#$ is a free S-module of rank 2.
We define, for an R-module M
$$M^\# = \operatorname{syz}_1^{R^\#}(M)$$

If M is MCM as R-modules, then $M^\#$ is MCM as $R^\#$-module
Also define, for $R^\#$-module N:

$$N^b = N/zN$$

If N is MCM over $R^\#$, N^b is MCM over R.

__Rmk__ $(R')^\#$ is $syz_1^{R^\#}(R)$, which is computed from

$$0 \to R^\# \xrightarrow{z} R^\# \to R \to 0, \text{ so}$$

$$R^\# = R^\#.$$

Similar for $(R^\#)^b$.

What does $\#$ and b do to MFs?

__Prop__ Let $M = \mathrm{cok}(\varphi,\psi)$ (i.e. (φ,ψ) is a MF and $M = \mathrm{coker}\,\varphi$)
Then the resolution of M over $S[[z]]$ is ($\tilde{F}, \tilde{G}$ are lifts of F, G to $S[[z]]$):

$$\begin{tikzcd}
0 & M \arrow[l] & \tilde{F} \arrow[l, "\pi"'] & & \tilde{G}\oplus\tilde{F} \arrow[ll, "{[\varphi \ zI]}"'] \arrow[dl] & & \tilde{F}\oplus\tilde{G} \arrow[ll, "{\begin{bmatrix} \psi & -zI \\ zI & \varphi \end{bmatrix}}"] \\
& & & M^\# \arrow[ul] \arrow[dl] & & & \\
& & 0 & & 0 \arrow[ul] & &
\end{tikzcd}$$

So the matrix factorization for $M^\#$ is $\left(\begin{bmatrix} \psi & -zI \\ zI & \varphi \end{bmatrix}, \begin{bmatrix} \varphi & zI \\ -zI & \psi \end{bmatrix}\right)$

__Cor__ $syz_1^{R^\#}(M^\#) = \mathrm{cok}\begin{bmatrix} \varphi & zI \\ -zI & \psi \end{bmatrix} \cong \mathrm{cok}\begin{bmatrix} \psi & -zI \\ zI & \varphi \end{bmatrix} = M^\#.$

Convergence: If $M = \mathrm{cok}(\varphi,\psi)$, then $(M^\#)^b = \left(\mathrm{cok}\begin{bmatrix} \varphi & -zI \\ zI & \psi \end{bmatrix}\right)^b =$

$$= \mathrm{cok}\begin{bmatrix} \psi & 0 \\ 0 & \varphi \end{bmatrix} = syz_1^R(M) \oplus M$$

To consider the other direction $(N^\flat)^\#$, where N is a MCM $R^\#$-module, recall that $R^\#$ is a free S-module (of rank 2) and therefore N is also free over S.
Let φ be the matrix over S representing multipl. by z: $N \xrightarrow{z} N$

Prop ① The matrix factorization for $N^\flat$ is $(\varphi, -\varphi)$
② If we consider $zI \pm \varphi$ a matrix over $S[[z]]$, then $N = \operatorname{cok}(zI - \varphi, zI + \varphi)$

Proof ① $(\varphi)(-\varphi) = -\varphi^2 = -z^2 I = fI$ on N.
② Similar. □

Cor If $\frac{1}{2} \in S$ then $(N^\flat)^\# = N \oplus \operatorname{syz}_1^{R^\#}(N)$ for a MCM $R^\#$-module N.

Proof $(N^\flat)^\# = (\operatorname{cok}(\varphi, -\varphi))^\# = \operatorname{cok}\left(\begin{bmatrix} -\varphi & zI \\ zI & \varphi \end{bmatrix}, \begin{bmatrix} \varphi & zI \\ -zI & -\varphi \end{bmatrix}\right)$

$$\cong \operatorname{cok}\begin{pmatrix} zI - \varphi & 0 \\ 0 & zI + \varphi \end{pmatrix}$$

row + column ops + divide by 2!

Thm (Knörrer) If $R^\#$ has only finitely many (or countably many ...) isom. classes of indec. MCM modules then so has R.
If $\frac{1}{2} \in S$, the converse holds as well.

<u>Proof</u> (1^{st} statement) Let $N_1, \dots, N_t$ be the indec. MCM $R^\#$-modules. Write

$$N_i^b = \bigoplus_{j=1}^{n_i} M_{ij}$$

a decomposition of N^b into indec. R-modules.

<u>Claim</u>: Every indec. MCM R-module is among the M_{ij}.

Let M be one. Then

$$M^\# = N_1^{e_1} \oplus \dots \oplus N_t^{e_t}$$

(Krull-Schmidt), so

$$\begin{array}{c} M^{\#b} = (N_1^{e_1})^b \oplus \dots \oplus (N_t^{e_t})^b \\ \| \\ M \oplus syz_1^R(M) \end{array}$$

So (KS again) M is $\oplus$-and of the RHS, so is one of the M_{ij}. $\square$

What about using z^3 (or z^r, $r \geq 3$)?

<u>Prop</u> (Herzog-Popescu '97) Let $R = S/(f)$ and $R' = S[[z]]/(f+z^r)$. Then for a MCM R'-module N

$$syz_1^{R'}(N/z^{r-1}N) \cong N \oplus syz_1^{R'}(N).$$

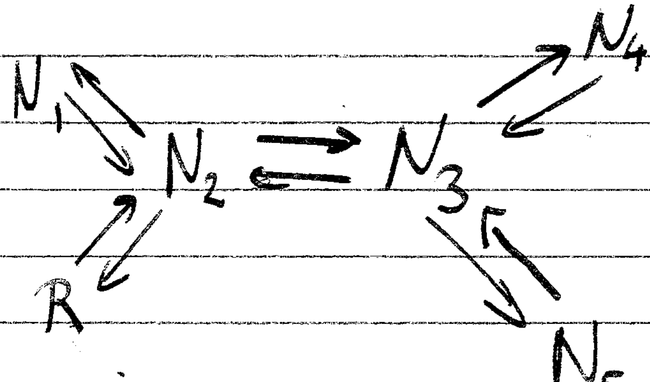
<u>Side note</u>: We have $(M^\#)^b = M \oplus syz_1^R(M)$
$(N^b)^\# = N \oplus syz_1^{R^\#} N$

But it's not clear whether the "splitting" happens on the way up or down. It depends on the module:

<u>e.g.</u>: D_5-hypersurface, $d=2$

$$x^2y + y^4 + z^2 = 0$$

Thus defines $R^\#$, where $R = k[[x,y]]/(x^2y+y^4)$
There are 6 indec. MCM $R^\#$-modules: the AR-quiver



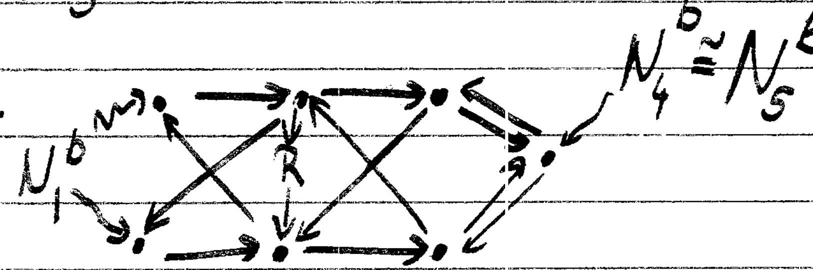
$$N_1 = \mathrm{cok}\left(\begin{bmatrix} z & x^2+y^3 \\ -y & z\end{bmatrix}, \begin{bmatrix} z & -x^2-y^3 \\ y & z\end{bmatrix}\right) \qquad N_1^b = \mathrm{cok}\begin{bmatrix}0 & x^2+y^3\\ -y & 0\end{bmatrix}$$

$$N_4 = \mathrm{cok}\left(\begin{bmatrix} z-iy^2 & x \\ -xy & z+iy^2\end{bmatrix}, \begin{bmatrix} & \\ & \end{bmatrix}\right) \qquad = \mathrm{cok}(y)\oplus\mathrm{cok}(x^2+y^3)$$

$$N_5 = \mathrm{cok}\left(\begin{bmatrix} z-iy^2 & xy \\ -x & z+iy^2 \end{bmatrix}, \begin{bmatrix} & \\ & \end{bmatrix}\right)$$

$$N_4^b = \mathrm{cok}\begin{bmatrix}-iy^2 & x\\ -xy & iy^2\end{bmatrix} \cong N_5^b.$$

The AR-quiver for R:



<u>The double double branched cover</u>

Set $R^{\#\#} = S[[z,w]]/(f+z^2+w^2)$
Then if $M = \mathrm{cok}(\varphi,\psi)$ is an R-module

$$(M^\#)^\# = \left(\mathrm{cok}\begin{pmatrix}\psi & -zI\\ zI & \varphi\end{pmatrix}\right)^\# = \mathrm{cok}\begin{pmatrix}\varphi & -zI & -wI & 0\\ zI & \psi & 0 & -wI \\ wI & 0 & \psi & zI \\ 0 & wI & -zI & \varphi\end{pmatrix}$$

This always decomposes (i.e. $\cong \mathrm{cok}\left(\begin{array}{c|c} \ast & 0 \\ \hline 0 & \ast \end{array}\right)$)

Let us be smarter: $R^{\#\#} \cong S[|u,v]]/(f+uv)$

$u = z+iw$

$v = z-iw$

and define $M^{\mathbb{X}} = \mathrm{cok}\left(\begin{bmatrix} \varphi & -vI \\ uI & \psi \end{bmatrix}, \begin{bmatrix} \psi & vI \\ -uI & \varphi \end{bmatrix}\right)$

__Thm__ (Knörrer) The funcdor $M \mapsto M^{\mathbb{X}}$ induces an equivalence of stable categories

$$\underline{MCM}(R) \simeq \underline{MCM}(R^{\#\#})$$

Just a few words about the proof of Knörrer's thm.

__"Cor"__: There is a bijection between the indec. MCM modules over R and $R^{\#\#}$.

In fact, Knörrer proves this first using

__Lemma__ If ${}_RM$ is indec. MCM, then

$$M^{\#} \text{ is decomposable} \iff \mathrm{syz}_1^R(M) \cong M$$

In this case, $M^{\#} \cong N \oplus \mathrm{syz}_1^{R^{\#}}(N)$ for some indec. $R^{\#}$-module N.
In particular, $M^{\#\#}$ always decomposes into precisely 2 indecomposable $\oplus$-ands.

Then Knörrer proves $\underline{\mathrm{Hom}}(M_1, M_2) \cong \underline{\mathrm{Hom}}(M_1^{\mathbb{X}}, M_2^{\mathbb{X}})$
(mostly just matrix multiplication)

[$\underline{MCM}(R)$: ("stable cat.") • objects MCM R-modules

7

• morphisms $\underline{\mathrm{Hom}}(M_1,M_2) \stackrel{\text{def}}{=} \mathrm{Hom}_R(M_1,M_2) / \{ f: M_1 \to M_2 \text{ factoring through free} \}$]

Classification of hypersurfaces of finite CM type
= only finitely many indec MCM

Of course it is ~~the~~ ADE-theorem:

Strategy (Buchweitz - Greuel-Schreyer '87) Use Knörrer periodicity to reduce to dim 1 (or 2) by showing that if $R = k[[x_0, \dots, x_d]]/(f)$ (k alg. closed, char $\neq 2$) has finite CM type, then

$$R \cong k[[x,y,z_2,\dots,z_d]]/(g(x,y)+z_2^2+\dots+z_d^2)$$

where $k[[x,y]]/(g(x,y))$ is an ADE hypersurface ring, i.e. (in char 0), $g(x,y)$ is one of

(A_n) x^2+y^{n+1}
(D_n) x^2y+y^{n-1}
(E_6) x^3+y^4
(E_7) x^3+xy^3
(E_8) x^3+y^5

FACT The ADE hypersurfaces (in dim 1) indeed have finite CM type.

BTW, the same results holds for countable CM type (over $\mathbb{C}$), with $g(x,y)$ one of (A_∞) x^2
(D_∞) x^2y.

Def say a hypersurface $R = S/(f)$ is <u>simple</u> if the set $c(f) := \{\text{ideals } I \subseteq S \mid f \in I^2\}$ is finite.

<u>Thm</u> (BGS) If $R = S/(f)$ has finite CM type, then it is simple.

<u>Proof</u> Let $M = \operatorname{cok}(\varphi, \psi)$ be a MCM R-module. Then

$$fI = \varphi\psi$$

$$\Rightarrow f \text{ is in the square of } I_1(\varphi) + I_1(\psi) =: J(M)$$

$$\Rightarrow J(M) \in c(f).$$

On the other hand, if $J \in c(f)$, say $f = \sum_{i=1}^{r} x_i y_i$ for $x_1, \dots, x_r, y_1, \dots, y_r \in J$.

Then we can define a matrix factorization (φ, ψ), so that

$$I_1(\varphi) + I_1(\psi) = (\underline{x}, \underline{y})$$

Inductively: if $r = 1$, take $(\varphi_1, \psi_1) = (x_1, y_1)$

if $r \geq 2$, take $\left(\begin{bmatrix} x_r I & \varphi_{r-1} \\ \psi_{r-1} & -y_r I \end{bmatrix}, \begin{bmatrix} y_r I & \varphi_{r-1} \\ \psi_{r-1} & x_r I \end{bmatrix} \right)$

Since we assume that R has finite CM type,

$(\varphi_r, \psi_r) =: (\varphi, \psi)$ is a direct sum of copies of some finite list of indec. MFs

$$(\sigma_1, \tau_1), \dots, (\sigma_e, \tau_e).$$

Then $J(\varphi, \psi) = \sum_{i=1} J(\sigma_i, \tau_i)$.

So $J(\varphi, \psi)$ is a sum of a finite subset of the finite list of ideals coming from (σ_i, τ_i), so the number of J's is finite □

Rmk We may assume $J=(\underline{x},y)$ by throwing in $z \cdot 0$ as a summand of f, z a generator of I.

Prop [simplicity restricts the equation] Assume $R=S/(f)$ is simple ($f \in \mathfrak{n}^2$). We need $k=\bar{k}$.

1. R is reduced
2. $e(R) \leq 3$
3. If $d \geq 2$, then $e(R)=2$

If $f \in k[[x_0,\dots,x_d]]$ has order m, (equiv $e(k[[x]]/(f))=m$) $\mathrm{char}(k) \nmid m$, then there is a change of variables

$$f = u\,(x_d^m + b_2 x_d^{m-2} + \dots + b_{m-1}x_d + b_m)$$

no x_d^{m-1}

by Weierstrass preparation, where $b_i \in k[x_0,\dots,x_{d-1}]$

In particular, if $m=2$, we may put f in the form

$$f = x_d^2 + \underbrace{g}_{\text{no } x_d}$$

In particular $R=A^{\#}$, $A=k[[x_0,\dots,x_{d-1}]]/(g)$

We know that also A has finite CM type.

Repeat: Get $R \cong k[[x,y,z_2,\dots,z_d]]/(g(x,y)+z_2^2+\dots+z_d^2)$ and $k[[x,y]]/(g)$ has finite CM type.

The $g(x,y)$ that give finite CM type in dim 1 were classified by
- Drozd-Roiter '70s (with some assumptions)
- Greuel-Knörrer ~80s

• B-G-S

You could also stop one step earlier and classify $k[[x,y,z]]/(g(x,y)+z^2)$ having finite CM type

<u>Thm</u> (Auslander): Let A be a complete local CM normal domain of dim 2, over an alg. closed field of char 0. If A has finite CM type, then
$$A \cong k[[u,v]]^G,$$
where $G \subseteq GL_n(k)$ is a finite group.
In particular if A is Gorenstein, then $G \subseteq SL_2(k)$ and so A is a Kleinian singularity!