Matrix factorization seminar 28/02/13

Jürgen Herzog "Linear maximal CM modules over strict complete intersections", 1981 (Backelin, H., Ulrich)

[linear maximal CM modules $\hat{=}$ Ulrich modules]

(R, m, k) Noetherian CM, M MCM
Then we always have: $\mu(M) \leq e(M)$ = multiplicity
min. number of gen
M is Ulrich-module if $=$.

Open question: does any such R admit Ulrich modules?

Thm 1 Suppose R admits an Ulrich module. Let $f \in m$, s.t. f^* = leading term of f in $gr_m(R)$ is a non-zerodivisor. Then R/fR admits an Ulrich module.

Cor If R is strict complete intersection $\Rightarrow$ R has an Ulrich module.

Thm 2 Let $I \subset R$ noetherian, $d > 1$ an integer, $f \in I^d$. Then there exists an integer $m \geq 1$ and square matrices $\alpha_1, \dots, \alpha_d$ s.t. (1) $f E_m = \alpha_1 \cdots \alpha_d$ ($m \times m$ mat.)
(2) $\sum_i \underbrace{I_1(\alpha_i)}_{\text{ideal gen. by the entries of } \alpha_i} = I$.

<u>Def</u> A sequence $\alpha = (\alpha_1, \dots, \alpha_d)$, $d \geq 1$ of square matrices of size m with coeff's in R is called a <u>matrix factorization</u> of f if

$$fE_m = \alpha_1 \cdots \alpha_d = \alpha_2 \alpha_3 \cdots \alpha_d \alpha_1 = \dots = \alpha_d \cdots$$

cyclic

<u>Thm 3</u> Let $f \in I^d$. Then there exists a matrix factorization $\alpha = (\alpha_1, \dots, \alpha_d)$ of f with $\sum_i I_1(\alpha_i) = I$.

<u>Special case</u> k field, $S = k[x_1, \dots, x_n]$, f homogeneous of degree d.

Want a matrix factorization of f $\alpha = (\alpha_1, \dots, \alpha_d)$ with the entries of the α_i linear forms.

$V = \mathrm{Hom}_k(S_1, k)$, $e_1, \dots, e_n$ be the dual basis of $x_1, \dots, x_n$

Let V_1, V_2 be k-vector spaces with given bases $(g_1, \dots, g_m)$, $(h_1, \dots, h_m)$, $\gamma = (\gamma_{ij})$ $m \times m$ matr. of linear forms

$$\phi_\gamma : V \to \mathrm{Hom}_k(V_1, V_2)$$

$$x \mapsto \phi_k(x) \quad \text{with } \phi_k(x)(g_k) = \sum_{l=1}^{m} \gamma_{kl}(x) h_l$$

$x \in V$, $x = \sum x_i e_i$

$m \times m$ matrices with linear forms as entries are in bijection to k-linear maps $V \to \mathrm{Hom}_k(V_1, V_2)$.

$\alpha = (\alpha_1, \dots, \alpha_d) \leftrightsquigarrow \phi_i : V \to \mathrm{Hom}_k(M_i, M_{i+1})$

where $M = \bigoplus_{i=1} M_i$

[$\vec{\alpha}$ and M given] $\dim M_i = m \ \forall i$

identify $M_{d+1} = M_1$

$$\Rightarrow f(x)\, \mathrm{id}_{M_1} = \phi_d(x) \circ \dots \circ \phi_1(x) \quad (*)$$

$(*) \Leftrightarrow f(x) = \alpha_1 \cdots \alpha_d$.

We consider M as $\mathbb{Z}/d\mathbb{Z}$-graded module over the tensor algebra $T = T(V)$: $x \in V$, $m \in M_i \Rightarrow xm = \phi_i(x)m$

$$\Rightarrow \underbrace{(x \otimes \cdots \otimes x)}_{d\text{-times}}(m) = (\phi_{i+d-1}(x) \circ \cdots \circ \phi_i(x))(m)$$
$$= f(x)m.$$

Define $I(f) = (x \otimes \cdots \otimes x - f(x)) \subset T$ two-sided ideal

Then $C(f) = T(V)/I(f)$ is called the <u>universal (generalized) Clifford algebra of f</u>. (cf. Roby '69)

The $\mathbb{Z}/d\mathbb{Z}$-graded $C(f)$-module M with $\dim_k M < \infty$ is called a Clifford module of f.

<u>Thm</u> Let $f \neq 0$, homog. form of degree d. Then the equivalence class of matrix fact of f correspond bijectively to Clifford modules of f.

<u>Thm</u> (Backelin, H., Sanders '86, char $> d$, Childs)
(a) The natural inclusion map $V \to T(V)$ induces an injective map $V \to C(f)$. ($\Rightarrow C(f) \neq 0$)
(b) $\dim_k C(f) < \infty \Leftrightarrow n = 1, d = 2$.

$f \in k[x_1, \dots, x_m]$, $g \in k[x_{m+1}, \dots, x_n]$

Matrix fact of $f+g$?

$C(f)\hat{\otimes}C(g)$: $(a\otimes b)(c\otimes d) = \xi^{(\deg b)(\deg c)}\, ac\otimes bd$, $\quad \xi \in k$,
where ξ is a d^{th} root of unity

Denote by $V = \left(\bigoplus_{i=1}^{m} kX_i\right)^*$, $W = \left(\bigoplus_{i=m+1}^{n} kX_i\right)^*$

Then $C(f+g)$ is generated over k by $V+W$
$C(f)\hat{\otimes}C(g)$ -"- -"- k by $V\otimes 1 \oplus 1\otimes W$

Thm The map $V\oplus W \to V\otimes 1\oplus 1\otimes W$ induces an epimorph.
$$C(f+g) \to C(f)\hat{\otimes}C(g)$$
of k-algebras.

Proof $(x+y)^d = f(x+y)$ in $C(f)\hat{\otimes}C(g)$
Let $\alpha = (\alpha_1,\dots,\alpha_d)$ be a matrix fact. for f of size m
-"- $\beta = (\beta_1,\dots,\beta_d)$ -"- for g of size l.
M, N corresp. Clifford modules.
$\to M\hat{\otimes}N$ is a $\mathbb{Z}/d\mathbb{Z}$-graded module over $C(f)\hat{\otimes}C(g)$, which may be viewed as a Clifford module over $C(f+g)$.

Let γ be the matrix fact. corresp. to $M\hat{\otimes}N$.
The size of γ is $\dim_k(M\hat{\otimes}N)/d = (\dim_k \overset{m\cdot d}{M})\cdot(\dim_k \overset{l\cdot d}{N})/d$
$= m\cdot l\cdot d$.

Lemma: The generic form $g = \sum_{i=1}^{s}\prod_{j=1}^{d} X_{ij}$ has a matrix fact of size d^{s-1} with $S = \mathbb{Z}[\xi][X_{ij}]$
s.t. (1) all entries of the α_i are of the form $\xi^k X_{ij}$ or 0.

(2) $\sum_i I(\alpha_i) = (Y_{ij})$

<u>Proof by induction</u> : $g = \prod_{j=1}^{d} X_{ij}$ $\quad g = \tilde{g} + \prod_{j=1}^{d} X_{sj}$

$\tilde{\beta} \quad (X_{s1}, \dots, X_{sd})$

$$\Rightarrow \beta_\ell = \begin{pmatrix} \tilde\beta_{\ell-1} & \xi^{\ell-1} X_{s1} & & & \\ & \tilde\beta_{\ell-2} & \xi^{\ell-2} X_{s2} & & \\ & & \ddots & \ddots & \\ & 0 & & \tilde\beta_{\ell-d+1} & \xi^{\ell-d-1} X_{s,d-1} \\ \xi^{\ell-d} X_{sd} & & & & \tilde\beta_{\ell-d} \end{pmatrix} \quad \text{in } Q(\xi) \quad \xi \longrightarrow \underbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ a_{d,} & \dots & a_{dd} \end{pmatrix}}_{A} \quad \gamma = \varphi(d)$$

<u>Problem</u> : Given $f \in k[x_1, \dots, x_n]$ of degree a. What is the smallest possible rank of a Clifford module of f_a?

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<u>Bernd Ulrich</u> Criteria for Gorensteinness

<u>Thm</u> [Rego '78] R local ring of a reduced curve $/\mathbb{C}$.
R is Gorenstein if $\mathrm{Ext}^1_R(\mathcal{C}_R, R) = 0$.

$\mathcal{C}_R$: conductor

R local CM: R is Gorenstein $\Leftrightarrow$ $\mathrm{Ext}^i_R(M, R) = 0$ $\forall 1 \le i \le \dim R$ $\forall$ MCM M.

<u>Thm</u> [U. '84] R local CM, $\dim R = d$. M R-module, f. g, with rank $\dim M = d$

$$\mu(M) > \frac{\operatorname{rank}(M) - e(R)}{2}$$

R is Gorenstein if $\operatorname{Ext}^i_R(M,R) = 0 \quad \forall 1 \le i \le d$.
[iff if M is CM].

Notice: $\operatorname{rank} M \cdot e(R) = e(M)$

If (R,m) is any noeth local ring, $|R/m| = \infty$, $d = \dim R$

Take $J = $ (d general element of m), M MCM R-mod.

Then $e(M) = e_m(M) \underset{\text{gen. mult. th.}}{=} e_J(M) \underset{M \text{ MCM}}{=} \ell(M/JM) \geq \ell(M/mM)$

$\| $ Nakay.
$\mu(M)$

$\textcircled{=}$ holds iff $mM = JM$

Then M <u>maximally generated MCM</u> (= MGMCM) (or Ulrich mod)

<u>2nd motivation</u>: Lech's conjecture

Let $R \to S$ be a flat local map of Noeth. local rings:

Then $e(R) \le e(S)$? [Lech '60]

[still open!] [Lech proved several cases]

Some reductions: reduce to case $\dim R = \dim S$, R domain

<u>Prop</u> [Hochster) Lech's conjecture is true whenever R admits an MGMCM M.

<u>Proof</u> $\underset{\substack{\| \\ e(R)\cdot \operatorname{rk}_R(M)}}{e(M)} = \mu(M) = \mu_S(S \otimes_R M) \le \underset{\substack{\| \\ e(S)\cdot \operatorname{rk}_R(M)}}{e_S(S \otimes_R M)}$ □

<u>Thm</u> [Hanes '94] Let R be a 3-dim. homogeneous k-algebra char $k = p$: R has a sequence M_n of MCM R-modules with $\lim_{n\to\infty} e(M_n)/\mu(M_n) = 1$.

3. Linear Resolutions

$(S,\eta) \twoheadrightarrow (R,m)$, S regular, M finite R-mod., $F_\bullet$ minimal free resolution of M $\mathcal{F}_j(F_i) = \eta^{j-i} F_i$
$F_\bullet$ is a linear resolution if $gr_{\mathcal{F}}(F_\bullet)$ is acyclic resolution.

<u>Prop</u> [Brennan, Herzog, U. '87] If M is MCM R-mod.
Then M MGMCM $\Leftrightarrow$ M has a linear resolution.
In this case $gr_m(M)$ is a MCM over $gr_m(R)$.

[in gen. open: $\exists$ f.g. MCM ??]

4. Existence

<u>Let</u> (R,m) be a local CM ring. MGMCM R-modules exist if ~~in the ...~~ (0) $\dim R = 1$ ($m^{e(R)-1}$ ✓, $\hat{R}$ reduced, $\bar{R}$ normal. $\mathcal{C}_R$)

(1) $\dim R = 2$, R homogen. domain
[Brennan, Herzog, U.]
(2) rings of minimal multiplicity [same ref]
(3) determinantal rings [max. minors: BHU, 2x2.: Hanes, general: Schreyer (not written yet!)]
(4) veronese rings [Eisenbud-Schreyer, Hanes]
(5) hypersurface rings [Backelin-Herzog] $\to$ strict c.i.
⋮

<u>Strict complete intersections</u> [Backelin-Herzog-U.]
Let (R, m) be a local CM ring, f R-regular
<u>Recall</u>: [Eisenbud '80]

$$\{\text{equivalence classes of m.f. } (\alpha_1, \alpha_2) \text{ of } f \text{ with entries in } m\}$$
$$\Big\updownarrow 1:1$$
$$\{\text{iso classes of MCM } R/f\text{-modules, } \operatorname{pdim}_R M < \infty \text{ without free summands}\}$$

<u>Prop</u> $\{$Equivalence classes of m.f. $(\alpha_1, \cdots, \alpha_d)$ of f of size s with entries in $m\}$
$$\Big\updownarrow 1:1$$
$\{$isom. classes of filtrations $(R/f)^{\oplus s} = F = U_0 \supset U_1 \supset \ldots \supset U_d = 0$, so that $U_i \subset m\, U_{i-1}$, $M_i = U_{i-1}/U_i$ MCM $R/(f)$-mod., $\operatorname{pdim}_R M_i < \infty$, $\mu(M_i) = s\}$

<u>Proof</u> $f \cdot 1_s = \alpha_1 \cdots \alpha_d$

free R-mod. of rk s.

$$\left\{\begin{array}{lcl} G = R^s & \twoheadrightarrow & F \\ \cup & & \\ V_1 = \alpha_d(G) & \twoheadrightarrow & U_1 \\ \cup & & \\ V_2 = \alpha_{d-1}(V_1) & \twoheadrightarrow & U_2 \\ \cup & & \\ \vdots & & \\ \cup & & \\ V_d = f \cdot G & \longrightarrow & U_d = 0 \end{array}\right.$$

$U_i \subset m U_{i-1}$ clear. $M_i = U_{i-1}/U_i = V_{i-1}/V_i \Rightarrow$

min. free res: $0 \to R^s \xrightarrow{\alpha_{d+1-i}} R^s \to M_i \to 0$
$\Rightarrow M_i$ is MCM over R/f.
number of gen's: clear. $\square$

<u>Thm</u> (R,m) local CM ring, N MCM R-module, $f \in m^d$ R-regular. There exists an $R/(f)$-module M such that $N \otimes_R M$ MCM $R/(f)$-module and

$$e(N \otimes_R M) \leq \frac{e(N/fN)}{d} \cdot \mu(M).$$

<u>Proof</u> f has a m.f. $(\alpha_1, \dots, \alpha_d)$ of size s with entries in m
$\Rightarrow$ filtration $R/(f)^s = F \supset U_1 \supset \dots \supset U_d = 0$ $\quad M_i = U_{i-1}/U_i$

$$0 \to R^s \xrightarrow{\alpha_{d+1-i}} R^s \to M_i \to 0 \quad / \otimes_R N$$
$$0 \to N \otimes_R R^s \to N \otimes_R R^s \to N \otimes_R M_i \to 0 \text{ exact}$$
[because: $\det(\alpha_{d+1-i})$ are NZD's in ring $\Rightarrow$ inj ✓]

$\Rightarrow N \otimes_R M_i$: MCM $R/(f)$-module [depth lemma]
and $\mathrm{Tor}_1^R(M, M_i) = 0$
$\Rightarrow N \otimes_R F \supset N \otimes_R U_1 \supset \dots \supset N \otimes_R U_d = 0$ filtration with factors $N \otimes_R M_i$
$\Rightarrow e(N \otimes_R F) = \sum_{i=1}^{d} e(N \otimes M_i) \geq d \cdot e(N \otimes M)$ for some M_i
$\quad \| $
$e(N/fN) \cdot \mu(M)$
$\square$

<u>Cor</u> (R,m) local CM, $f \in m$ whose leading form f^* in $gr_m(R)$

is regular. If R has a MGMCM then so does $R/(f)$.

Proof $f \in \mathfrak{m}^d \setminus \mathfrak{m}^{d+1} \Rightarrow f^*$ has degree d.
Let N MGMCM R-module
$\Longrightarrow gr_{\mathfrak{m}}(N)$ is still MCM $gr_{\mathfrak{m}}(R)$-mod.
$\Longrightarrow f^*(R)$ is an $gr_{\mathfrak{m}}(R)$
$\Longrightarrow e(N/fN) = d \cdot e(N) = d \cdot \mu(N)$
$\Longrightarrow N \otimes_R M$ is MGMCM $R/(f)$-module. $\square$.