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# Matrix factorization seminar

David Eisenbud: Matrix factorizations in higher codim =  
Lecture (joint w/ Steve Prace

- |                                 |                             |
|---------------------------------|-----------------------------|
| 0. Intro/Goals                  | 6. The intermediate modules |
| 1. Definition                   | 7. Infinite Resolutions     |
| 2. High Syzygies                | 8. Decomposition of Ext     |
| 3. Example                      | 9. Functoriality            |
| 4. Finite resolution            | 10. Problems                |
| 5. Proof of exactness - codim 2 |                             |

Setting:  $S$  regular local ring,  $R = S/(f_1, \dots, f_c)$ , CI of codim  $c$   
Codim 1:  $S$  infinite residue field

$$mf \quad A_1 \xrightarrow{d} A_0 \xrightarrow{h} A_1 \xrightarrow{d} A_0 \quad / S$$

Codim c  $mf$  is  $(d, R)$   $A_1 \xrightarrow{d} A_0$   
 $A_n$  filtered  $A_i = A_i(c) \supset \dots \supset A_i(p) \dots A_i(1)$

$$\bigoplus A_i(p) \xrightarrow{h} A_1 \xrightarrow{d} A_0 \quad \text{preserves filtration} \quad \text{free } \oplus$$

$$\bigoplus_{g \leq p} A_0(g) \xrightarrow{(h_1, \dots, h_p)} A_1(p) \xrightarrow{d_p} A_0(p) \xrightarrow{f_p} A_1(p) \rightarrow A_0(p) \text{ mod } (f_1, \dots, f_{p-1})$$

$$B_1(p) = A_1(p)/A_1(p-1) \xrightarrow{f_p} B_1(p)$$

$$R(p) = S/(f_1, \dots, f_p) \quad M(p) = \text{coker } R(p) \otimes d_p$$

$$M = M(c) = \text{coker } R \otimes d \quad \text{"the mf module"}$$

Thm If  $R, S$  is as above,  $M$  is a sufficiently high  $R$ -syzygy, and if  $f_1, \dots, f_c$  are generators of  $(f_1, \dots, f_c)$  then  $M = M'(c)$  for some mf. set  $f_1, \dots, f_c$ .

3. Example → see copy

in general:  $\text{Ext}_R(M, k)$  is a f.g. module over  $k[x_1, \dots, x_c]$

$$\begin{array}{ccccccc} \dots & F_2 & \xrightarrow{\delta} & F_1 & \xrightarrow{\delta} & F_0 & \rightarrow M \\ & \downarrow & & \downarrow & & \downarrow & \\ & F_2 & \xrightarrow{\delta} & F_1 & \xrightarrow{\delta} & F_0 & \end{array} \quad R\text{-free}$$

$$\tilde{\delta}^2 \equiv 0 \quad (f_1, \dots, f_c)$$

$$\tilde{\delta}^2 = \sum t_i f_i \quad t_i \equiv \tilde{t}_i \pmod{(f_1, \dots, f_c)}$$

[What does sufficiently high mean? use CM with no free summands  
 ↓  
 (2c-1) steps beyond regularity

Minimal S-free resolution:  $0 \rightarrow (A, (1) \xrightarrow{d_1} A_0(1) \rightarrow M(1)$   
 $R(1) \otimes \left[ \begin{array}{c} B, (1) \\ \otimes S/f_1 = R(1) \end{array} \right]$

$B(p) = (B, (p) \xrightarrow{b_p} B_0(p))$   
 $\mathbb{L}(p-1)$  resolves  $M(p-1)$  over  $S$

**Example**

$$S = R[x, y, a, b]$$

$$R = S / (ax, by)$$

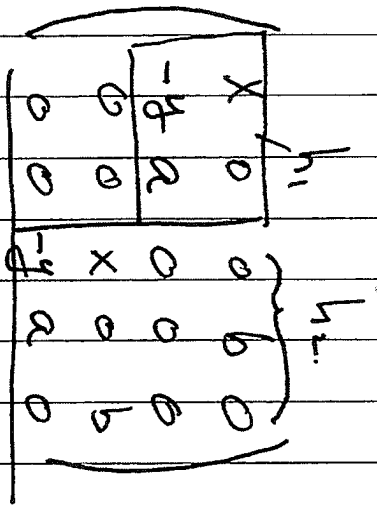
$$M = S_{y_2}^R (R(x, y))$$

$$d_1 h_1 = h_1, d_1 = ax$$

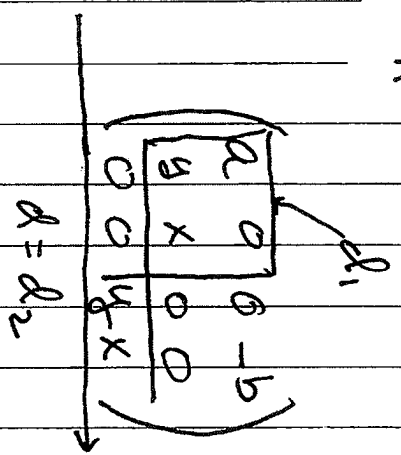
$$d_2 h_2 = \begin{pmatrix} by & 0 & 0 & 0 \\ 0 & by & 0 & 0 \\ 0 & ax & by & 0 \end{pmatrix} \equiv by \text{ mod } ax$$

$$h_2 d_2 = \begin{pmatrix} ax & by & 0 & 0 \\ 0 & 0 & -by & 0 \\ -ax & 0 & 0 & ax \\ 0 & 0 & 0 & by \end{pmatrix} \leftarrow \equiv by \cdot \text{projection mod } ax$$

$$\overline{A_1(2)} \oplus \overline{A_0(2)} = R^5$$



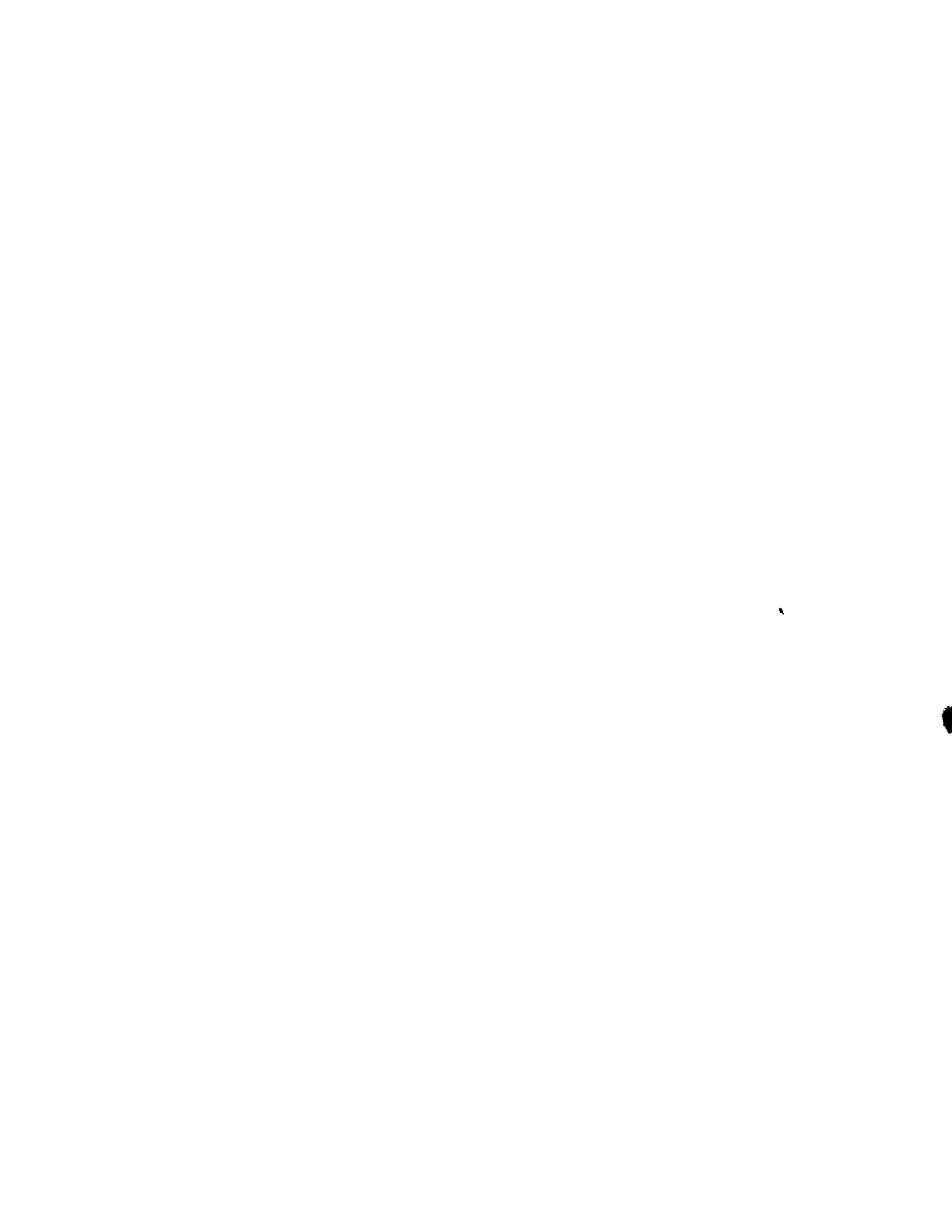
$$\overline{A_1(1)} = R^2 \cap \overline{A_1(2)} = R^4$$



$$\overline{A_0(1)} = R^2$$

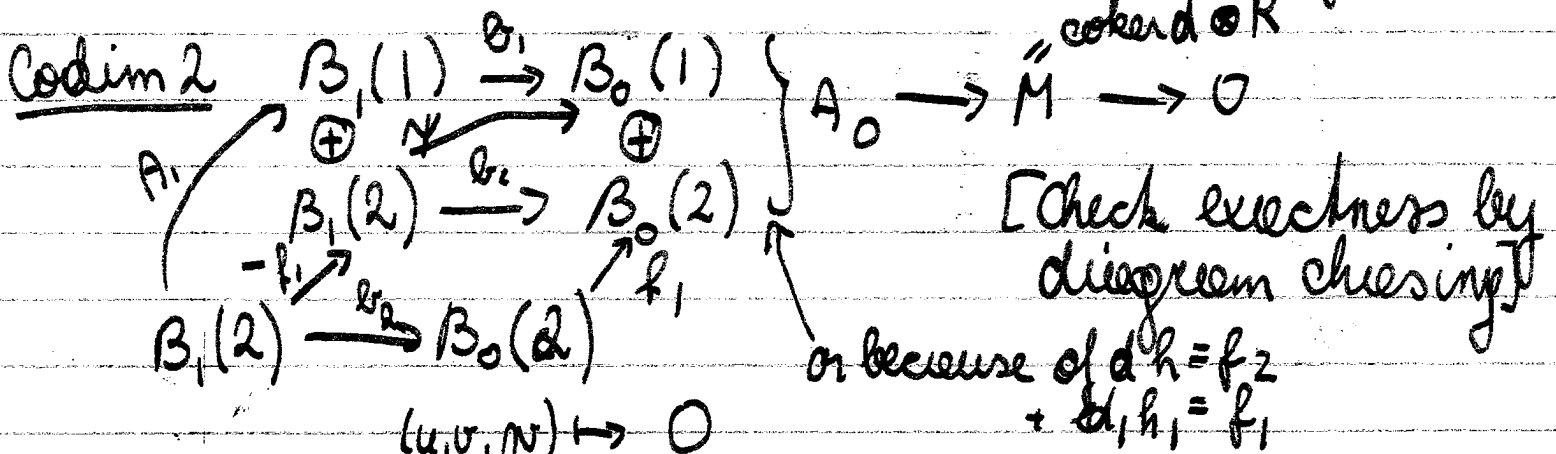
$$\overline{A_0(2)} = R^3 \cap \overline{A_0(1)} = R^2 \oplus R^1$$

(1)



(3)

$$0 \rightarrow \mathcal{L}(p-1) \rightarrow \mathcal{L}(p) \rightarrow B(p) \otimes K(f_1, \dots, f_{p-1}) \rightarrow 0$$



$$(u, v, w) \mapsto 0$$

Proof that it comes from  $B_1(2)$ :  
 $g_2(v) = -f_1 w$      $g_1(u) = -f_2 v$   
 $\pi_2 h_2 d_2(u, v) = f_2 \pi(u, v) \pmod{f}$   
 $= f_2 v$

$$-f_1(\pi_2 h_2(0, w))$$

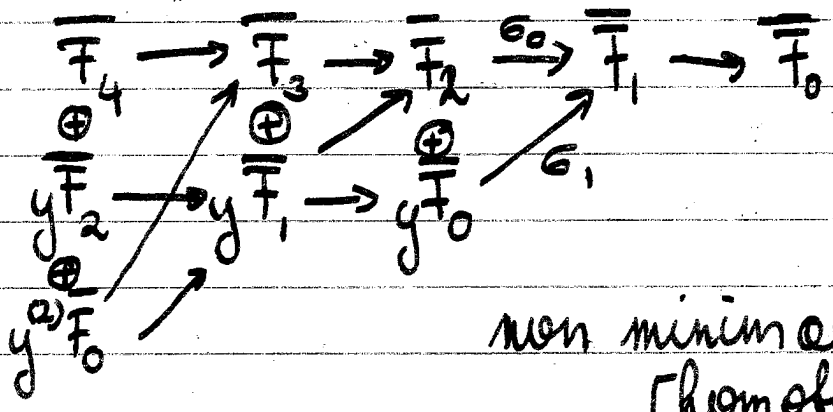
$$v = f_1 x, \quad \pi_2 h_2 w = f_2 x$$

Thus may add cycles  $(u, v, w) \equiv (u, 0, w) \pmod{\mathcal{O}}$   
 $g_1(u') = 0 \Rightarrow u' = 0$   
 $f_1 w = 0 \Rightarrow w' = 0$

Thm If  $M$  is  $\text{Syzy}_m^R N$  is an mf module and if  $\text{syzy}_m^{R(p)} N$  is MCM then  $M(p) = \text{syzy}_m^{R(p)} N$

Shenash construction:  $S \rightarrow S/f$   $f \neq 0$   
 $M$  an  $S/f$ -module,  $S$ -free resolution  
 $F_n \rightarrow \dots \rightarrow F_1 \xrightarrow{f} F_0 \rightarrow M \rightarrow 0$   
 $g_0 = \delta, g_1 = \text{homotopy for } f \text{ on } F_1$

$$\sum_{k+l=i} \sigma_k \sigma_l = 0 \quad \sigma_i : F \rightarrow F[l_i - 1]$$



non minimal resolution of  $M/S/f$   
[homologies not min].

Construction of a minimal  $R$ -free resolution of  $M$ -over  $R$  mod from a codim  $c$  m.f.  $(d, h)$ .

① Shereshevskii const.  $B_1(1) \xrightarrow[h_1]{d_1} B_0(1)$  gives  $\xrightarrow{d_1} \xrightarrow[h_1]{h_1} \xrightarrow{d_1}$  over  $R(1)$

(p)  $T(p-1) \rightarrow \dots \rightarrow T(p-1)_2 \rightarrow A_1(p-1) \xrightarrow{d_{p-1}} A_0(p-1)$   
 $\oplus \quad \searrow \quad \oplus$   
 $B_1(p) \xrightarrow{d_p} B_0(p)$

$\rightsquigarrow$  mapping cone  
 $(p-1) \text{ Sh(mapping cone, } \mathcal{J}) = T(p)$

Then: This minimal.

Theorem: If  $M$  is a m.f. module  $p \dots M(p) \dots$  exterior  
 $\text{Ext}_S(M(p), k) = \bigoplus_{i=1}^c \bar{B}(q) \otimes k \langle e_1, \dots, e_{q-1} \rangle$

$$\text{Ext}_{R(p)}(M(p), k) = \bigoplus_{i=1}^c \bar{B}(q) \otimes k[x_1, \dots, x_c]$$

Time for functors!

[as u.s.p. with some action]  $[M(0) = 0]$   
 $\bar{B}_1 \dots \rightarrow \bar{B}_1(p) = \bar{B}_1(p) \otimes k \oplus \bar{B}_0(p) \rightarrow \bar{B}_0(p) \otimes k$

$$A_0(p-1) \subset A_0(p) \rightsquigarrow R(p) \otimes M(p-1) \xrightarrow{m_p} M(p)$$

MF: MCM'(R)  $\longrightarrow$   $\Pi$  maps MCM  $(R(p))$

finite

MCM' = MCM that are m.f. v.f. finite.

$$R(p) \otimes M(p-1) \rightarrow M(p)$$

[gen. of periodicity in codim 1  $\rightsquigarrow$   $p=0$ :  $M = \text{Syz}_1 M$ ]  $\text{Syz}_2^{R(p)}(M(p))$

$$M = \text{Syz}_{n \gg 0}^{R(p)} N$$

$$R(p) \text{Syz}_m^{R(p)} N \xrightarrow{m_p} \text{Syz}_m^{R(p)} N$$

$$\text{Syz}_{m-2}^{R(p)} N$$

Take  $M$  module  $/R$  (e.g. MCM,  $xy \neq yxy, \dots$ )  
 $E = \text{Ext}(M, k)$

[stalled]  $\downarrow$

$$\text{Ext}(M, k) \xrightarrow{\alpha} H_*^0(\tilde{E}) = \bigoplus_{d \in \mathbb{Z}} H^0(\tilde{E}(d))$$

$$\bigoplus H_*^i(\tilde{E})$$

Is  $\alpha$  a surjection?  $\checkmark$

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(2) If  $\text{rank}_{\mathbb{R}} \text{Ext}_{\mathbb{R}}(M, k) \leq 1 \stackrel{?}{\Rightarrow} \text{Syz}_{\mathbb{R}} M$  is MCM?

(3) Some hypotheses  $\stackrel{?}{\Rightarrow} \text{Syz}_{\mathbb{R}} M$  is a m.f. module?  
(know:  $\text{Syz}_{\mathbb{Z}} M$  is m.f.)

(4) What happens in the non-local case?

(5) (Question for Reigner): factorization for higher codim

$$\begin{pmatrix} a_1 & x & b_1 \\ a_2 & y & b_2 \end{pmatrix}$$

$\leadsto$  What can one say about e.g. MCM's on non-complete intersections (e.g. the  $2 \times 2$  minors of a  $2 \times 3$  matrix)?  $\rightarrow$  get something from  $\mathbb{Z}$ -case?