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Matrix factorization seminar

__David Eisenbud__: Matrix factorizations in higher codimension (joint w/ Irena Peeva)

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2. High Syzygies
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__Setting__: S regular local ring, $R = S/(f_1, \dots, f_c)$, CI of codim c
S infinite residue field

__Codim 1__:

$$\text{mf} \qquad \begin{tikzcd} A_1 \arrow[r, "d"] \arrow[rr, bend right, "f"] & A_0 \arrow[r, "h"] \arrow[rr, bend left, "f"] & A_1 \arrow[r, "d"] & A_0 \end{tikzcd} \quad /S$$

__codim c__ mf is (d, h) $\quad A_1 \xrightarrow{d} A_0$

A_n __filtered__ $\quad A_i = A_i(c) \supset \dots \supset A_i(p) \dots A_i(1)$:
free ⊕

$$\oplus A_0(p) \xrightarrow{h} A_1 \xrightarrow{d} A_0 \quad \text{preserves filtration}$$

$$\oplus A_0(q) \xrightarrow{(h_1, \dots, h_p)} A_1(p) \xrightarrow{d_p} A_0(p)$$
$q \le p$

$$\begin{tikzcd} A_1(p) \arrow[r, "d_p"] \arrow[d, two heads] & A_0(p) \arrow[r, "h_p"] \arrow[rr, bend left, "f_p"] & A_1(p) \arrow[r] \arrow[d, two heads] & A_0(p) \\ B_1(p) = A_1(p)/A_1(p-1) \arrow[rr, "f_p"] & & B_1(p) & \end{tikzcd} \quad \text{mod } (f_1, \dots, f_{p-1})$$

$R(p) = S/(f_1,\dots,f_p)$ $\quad$ $M(p) = \operatorname{coker} R(p)\otimes d_p$

$M = M(c) = \operatorname{cok} R\otimes d$ $\quad$ "the mf module"

__Thm__ If R, S is as above, M is a sufficiently high R-syzygy, and if $f_1' \dots f_c'$ are generators of $(f_1,\dots,f_c)$ then $M = M'(c)$ for some mf. wrt $f_1',\dots,f_c'$.

3. __Example__ $\to$ see copy

__in general__: $\operatorname{Ext}_R(M,k)$ is a f.g. module over $k[x_1,\dots,x_c]$

$$\dots F_2 \xrightarrow{\delta} F_1 \xrightarrow{\delta} F_0 \to M \qquad R\text{-free}$$
$$\tilde F_2 \xrightarrow{\tilde\delta} \tilde F_1 \xrightarrow{\tilde\delta} \tilde F_0$$

$\tilde\delta^2 \equiv 0 \ (f_1,\dots,f_c)$

$\tilde\delta^2 = \sum \tilde t_i f_i \qquad t_i \equiv \tilde t_i \mod (f_1,\dots,f_c)$

[What does __sufficiently high__ mean? ws CM with no free summands

$\qquad$ $(2c-1)$ steps beyond Regularity]

__Minimal S-free resolution__: $0 \to R(1)\otimes \begin{cases} A_1(1) \\ B_1(1)\end{cases} \xrightarrow{d_1} A_0(1) \to M(1)$ $\qquad \otimes_{S/f_1} = R(1)$

$\mathbb{B}(p) = \qquad (B_1(p) \xrightarrow{b_p} B_0(p))$

$\mathbb{L}(p-1)$ resolves $M(p-1)$ over S

$\boxed{\text{Example}}$

$S = k[x, y, a, b]$ $\quad R = S/(ax, by)$ $\quad M = \mathrm{Syz}_2^R(R/(x,y))$

$d_1 h_1 = h_1 d_1 = ax$ $\quad d_2 h_2 = \begin{pmatrix} by & 0 & 0 \\ 0 & by & 0 \\ 0 & ax & by \end{pmatrix} \equiv by \bmod ax$

$h_2 d_2 = \begin{pmatrix} bx & bx & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ax & 0 & -by & 0 \\ 0 & ax & 0 & by \end{pmatrix} \longleftarrow \equiv by \cdot \text{projection} \bmod ax$

$$\begin{array}{c} \bar{A}_1(2) \\ \oplus \\ \bar{A}_0(2) \\ \| \\ R^5 \end{array} \xrightarrow{\left(\begin{array}{cc|ccc} x & 0 & 0 & b & 0 \\ -y & a & 0 & 0 & 0 \\ \hline 0 & 0 & x & 0 & b \\ 0 & 0 & -y & a & 0 \end{array}\right)} \begin{array}{c} \bar{A}_1(1) = R^2 \\ \cap \\ \bar{A}_1(2) \\ \| \\ R^4 \end{array} \xrightarrow[d = d_2]{\left(\begin{array}{cc|cc} a & 0 & 0 & -b \\ y & x & 0 & 0 \\ \hline 0 & 0 & y & x \end{array}\right)} \begin{array}{c} \bar{A}_0(1) = R^2 \\ \cap \\ \bar{A}_0(2) \\ \| \\ R^3 \end{array} \xrightarrow{\begin{pmatrix} -y & a & 0 \\ x & 0 & b \end{pmatrix}} R^2 \xrightarrow{(x \;\; y)} R^1$$

with $\left(\begin{smallmatrix} x & 0 \\ -y & a \end{smallmatrix}\right)$ labelled h_1, the columns $\left(\begin{smallmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ x & 0 & b \\ -y & a & 0 \end{smallmatrix}\right)$ labelled h_2, and $\left(\begin{smallmatrix} a & 0 \\ y & x \end{smallmatrix}\right)$ labelled d_1.

$R^3 \to M \to 0$

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$$0 \to \mathbb{L}(p-1) \to \mathbb{L}(p) \to \mathbb{B}(p) \otimes \mathbb{K}(f_1, \ldots, f_{p-1}) \to 0$$

<u>Codim 2</u>

$$\begin{tikzcd}
 & B_1(1) \arrow[r, "b_1"] \arrow[dr, "\psi"] & B_0(1) \\
 & \oplus & \oplus \\
 & B_1(2) \arrow[r, "b_2"] & B_0(2) \\
B_1(2) \arrow[uuur, bend left, "A_1"] \arrow[ur, "-f_1"] \arrow[r, "b_2"] & B_0(2) \arrow[ur, "f_1"] &
\end{tikzcd}$$

$\Big\}\ A_0 \to \overset{\text{coker } d \otimes R}{\overset{\shortparallel}{M}} \to 0$

[Check exactness by diagram chasing]

or because of $d^h = f_2$
$+ d_1 h_1 = f_1$

$(u, v, w) \mapsto 0$

Proof that it comes from $B_1(2)$:

$b_2(v) = -f_1 w \quad b_1(u) = -\psi v$

$\pi_2 h_2 \underbrace{d_2(u,v)}_{} = f_2 \pi(u,v) \mod f.$
$= f_2 v$

$-f_1(\pi_2 h_2(0, w))$

$v = f_1 x, \quad \pi_2 h_2 w = f_2 x$

Thus may add cycles $(u, v, w) \equiv (u, 0, w) \mod \partial$

$b_1(u') = 0 \Rightarrow u' = 0$

$f_1 w = 0 \Rightarrow w' = 0.$

<u>Thm</u> If M is $\mathrm{syz}^R_m N$ is an mf module and if $\mathrm{syz}_m^{R(p)} N$ is MCM then $M(p) = \mathrm{syz}_m^{R(p)} N$

<u>Shamash construction</u>: $S \to S/f \quad f$ nzd

M an S/f-module, S-free resolution

$$F_n \to \cdots \to F_1 \xrightarrow{\delta} F_0 \to M \to 0$$

$\sigma_0 = \delta, \ \sigma_1 =$ homotopy for f on F

$\sum_{k+l=i} \sigma_k \sigma_l = 0 \qquad \sigma_i : F \to F[li-1]$

$$\begin{tikzcd}
\overline{F}_4 \arrow[r] & \overline{F}_3 \arrow[r] & \overline{F}_2 \arrow[r, "\sigma_0"] & \overline{F}_1 \arrow[r] & \overline{F}_0 \\
\oplus & \oplus & \oplus & & \\
y\overline{F}_2 \arrow[r] \arrow[uur] & y\overline{F}_1 \arrow[r] \arrow[uur] & y\overline{F}_0 \arrow[uur, "\sigma_1"] & & \\
\oplus & & & & \\
y^{(2)}F_0 \arrow[uur] \arrow[uuuur] & & & &
\end{tikzcd}$$

non minimal resolution of $N/S/f$
[homotopies not min].

<u>Construction</u> of a minimal R-free resolution of M-coker $R\otimes$
from a codim c m.f. (d,h).

(1) Shenash constr. $B_1(1) \underset{h_1}{\overset{d}{\rightleftarrows}} B_0(1)$ gives $\xrightarrow{d_1}\xrightarrow{h_1}\xrightarrow{d_1}$ over $R(1)$

$$\begin{tikzcd}
(p) \ T(p-1) \arrow[r] & \dots \arrow[r] & T(p-1)_2 \arrow[r] & A_1(p-1) \arrow[r, "d_{p-1}"] & A_0(p-1) \\
& & & \oplus & \oplus \\
& & & B_1(p) \arrow[r, "b_p"] \arrow[uur, "\psi_p"] & B_0(p)
\end{tikzcd}$$

$\rightsquigarrow$ mapping cone

$(p{-}1) \quad Sh(\text{mapping cone}, J) = T(p)$

<u>Thm</u>: This minimal.

<u>Theorem</u>: If M is a mf module $p \dots M(p) \dots$ exterior

$$\mathrm{Ext}_S(M(p),k) = \bigoplus_{1}^{p} \overline{B}(q) \otimes k\langle e_1,\dots,e_{q-1}\rangle$$

$$\mathrm{Ext}_{R(p)}(M(p),k) = \bigoplus_{1}^{p} \overline{B}(p) \otimes k[x_q,\dots,x_c] \qquad [M(0)=0]$$

[as v.sp. with some action]

Time for functors!

$\overline{B}(\dots \to \overline{B}_1(p) = B_1(p)\otimes k \oplus \overline{B}_0(p) = B_0(p)\otimes k$

$$A_0(p-1) \subset A_0(p) \rightsquigarrow R(p)\otimes M(p-1) \xrightarrow{m_p} M(p)$$

MF: $\underline{MCM'(R)} \longmapsto \Pi$ maps $\underline{MCM}(R(p))$

$f_1,\dots,f_c$

$MCM' = MCM$ that are m f wrt. $f_1,\dots,f_c$.

$$\begin{tikzcd}
R(p)\otimes M(p-1) \arrow[r] & M(p) \arrow[d] \\
 & \operatorname{Cosyz}_2^{R(p)}(M(p)) \arrow[ul, "{[1]}"]
\end{tikzcd}$$

[gen. of periodicity in codim 1 $\rightsquigarrow p=0$: $M \cong \operatorname{cosyz}_2 M$]

$$M = \operatorname{Syz}^{R(c)}_{n \gg 0} N$$

$$\begin{tikzcd}
R(p)\ \operatorname{Syz}^{R(p-1)}_{m} N \arrow[r, "m_p"] & \operatorname{Syz}^{R(p)}_{m} N \arrow[dl] \\
\operatorname{Syz}^{R(p)}_{m-2} N \arrow[u, "{[1]}"] &
\end{tikzcd}$$

Take M module $/R$ (e.g MCM, syzygy, ...)

$$\begin{tikzcd}
E = \operatorname{Ext}(M,k) \arrow[d, "\text{[stabilised]}"'] \arrow[dr] & \\
\widehat{\operatorname{Ext}}(M,k) \arrow[r, "\alpha"] & H^0_*(\widetilde{E}) = \bigoplus_{d\in\mathbb{Z}} H^0(\widetilde{E}(d))
\end{tikzcd}$$

$\widehat{\operatorname{Ext}}(M,k) = \bigoplus H^i_*(\widetilde{E})$

Is α a surjection? ✓

(2) If $\operatorname{rep} \operatorname{Ext}_R(M,k) \leq 1 \overset{?}{\Rightarrow} \operatorname{syz}_1 M$ is MCM?

(3) Some hypotheses $\overset{?}{\Rightarrow} \operatorname{Syz}_1 M$ is a mf module?
(know: $\operatorname{Syz}_{2c} M$ is mf)

(4) What happens in the non-local case??

(5) (Question for Ragnar): factorization for higher codim)

$$\begin{pmatrix} a_1 & x & b_1 \\ a_2 & y & b_2 \end{pmatrix}$$

$\rightsquigarrow$ What are can one say about e.g MCM's on non-complete intersections (e.g. the 2×2 minors of a 2×3 matrix)? $\to$ get something from c.i. case??