

Algebra factorization seminar

19/02/2013

Rayner Buchweitz: about Eisenbud's paper.

"Homological algebra on a complete intersection with an application to group representations"
 TAMS 260, no. 1, (1980) 35-64.
 submitted Febr. 1978

Two rings: A, B $A \rightarrow B = A/I$; A commutative
 $I = (x_1, \dots, x_n)$, I/I^2 is free on the classes of the x_i as a basis.

Let M be a B -module and consider a free B -resolution

$$0 \leftarrow M \leftarrow F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \dots \xrightarrow{d_r} F_r \leftarrow \dots \leftarrow \mathbb{F}$$

Note: Can represent the differentials as matrices over B .
 Let each F_i be a free A -module \tilde{F}_i and each d_i be a morphism \tilde{d}_i of these free A -modules:

$$\mathbb{F} \cong 0 \leftarrow \tilde{F}_0 \xrightarrow{\tilde{d}_0} \tilde{F}_1 \xrightarrow{\tilde{d}_1} \dots \xrightarrow{\tilde{d}_r} \tilde{F}_r \leftarrow \dots$$

In general $\tilde{d}_i \neq 0$! But $\tilde{F}_i \otimes_A B = \mathbb{F}$, $\tilde{d}_i^2 \otimes_A B = 0$.
 $\Rightarrow \tilde{d}_i^2 = \sum_{j=1}^n x_j \tilde{t}_{ij}, \tilde{t}_{ij}: \mathbb{F} \rightarrow \mathbb{F}[2]$

Important fact: the $\tilde{t}_{ij} := \tilde{t}_{ij} \otimes_A B: \mathbb{F} \rightarrow \mathbb{F}[2]$

- are actual morphisms of \mathbb{F} .
- are unique up to B -homotopy
- are factorial up to homotopy

• the t_i, t_j commute up to homology.
[This is shown in the first section of the paper]

It is not relevant that \mathbb{F} was a resolution. It suffices that \mathbb{F} is a complex of B -free modules.

These facts can be summarized:

$$\tilde{D}(B) \cong K = K^-(\text{Free}(B))$$

f.g.

$$\begin{array}{c} \text{id}_K \\ \downarrow \\ t_i \in Z_{\text{gr}}^2(D^-(B)) \end{array}$$

[2]

Put these together and define: $Z_{\text{gr}}^i(D^-(B)) = \{z : \text{id} \rightarrow [i] \mid z \circ [1] = [(-1)^i z] [1]\}$

$$Z_{\text{gr}}^0(B) := \bigoplus Z_{\text{gr}}^i(D^-(B))$$

graded comm. ring under composition

We have a natural homomorphism of graded comm. rings [Note: small category]

$$B[t_1, \dots, t_n] \rightarrow Z_{\text{gr}}^0(B)$$

$|t_i| = 2$

The invariant description:

$$\text{Inm (Uchida?) : } B[t_1, \dots, t_n] = \text{Sym}_B \left(\text{Hom}_B(I/I^2, B) \right)$$

$N_{B/A}$: normal module of B w.r.t. A .

t_1, \dots, t_n : the dual basis to $x_1, \dots, x_n \text{ mod } I^2$.

ex: A regular local $I = (x_1, \dots, x_n) \subseteq m_A$: maximal ideal of A .

Serre-Vasconcelos: In this situation

$$I/I^2 \xrightarrow{\cong} (x_1, \dots, x_n) \text{ is an } A\text{-sequence}$$

$$(\Leftrightarrow) x_i \text{ is a NZD on } A, x_{i+1} \text{ is a NZD on } A/(x_1, \dots, x_i)$$

$$i = 1, \dots, n-1, \text{ and } (x_1, \dots, x_n)A \neq A$$

Study the d_i on a minimal resolution of B/m_B as B -module ($I = m_A^2$).

This construction was given by J. Tate (1958): ~~Take~~
 Assume that $m_A = (y_1, \dots, y_N)$ s.t. the y_j form an A -regular sequence. Take the Koszul ex. on the y_j :

$$K = K(y_1, \dots, y_N)A = \bigoplus_{v \geq 0} \wedge^v \left(\bigoplus_{i=1}^N A dy_i \right) [v] \text{ and}$$

$$\partial = \sum_{i=1}^N y_i \frac{\partial}{\partial y_i} \quad (\text{Euler derivation})$$

This is a minimal resolution of $k = A/m_A$ over A (Serre).

Now Tate: Take $G = \bigoplus A dx_i$ and form the divided power algebra $i=1$ on G :

$$T_i G := \text{Sym}_i (G^*)^*, \text{ where } * = \text{Hom}_A(-, A).$$

$$T G = \bigoplus_{i \geq 0} T_i G.$$

The point here is: $T_i G$ has a basis $dx_1^{(k_1)}, \dots, dx_n^{(k_n)}$, $\sum_j k_j = i$
 $x_i^{(k)} = \frac{x^k}{k!}$, $\frac{\partial x^{(k)}}{\partial x} = x^{(k-1)}$

Consider: $\tilde{\mathbb{H}} = \wedge^1 \left(\bigoplus_{j=1}^N A dy_j [1] \otimes_A T^0 \left(\bigoplus_{i=1}^N A dx_i [2] \right) \right)$

(4)

$$\tilde{\partial} = \sum_{j=1}^N y_j \frac{\partial}{\partial y_j} + \sum_{i=1}^n \sum_{j=1}^N x'_{ij} \frac{\partial}{\partial x_i} \quad \text{such that}$$

$$\sum_j x'_{ij} y_j = x_i \quad (x_i \text{'s unique modulo } I)$$

Facts: $\tilde{\partial}^2 = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \Rightarrow t_i$ correspond to $\frac{\partial}{\partial x_i}$.

Def: $\mathbb{F} = (\mathbb{F}, \tilde{\partial}) \otimes_A B$ is the (minimal resolution of B/m_B over B)

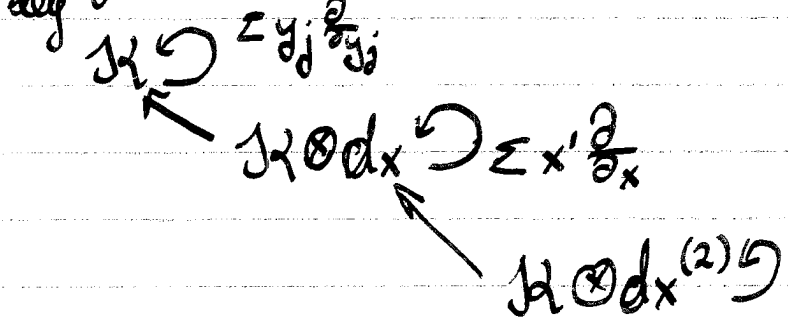
$$\tilde{\partial} = \sum_{i=1}^n x_i t_i$$

Now: $B[t_1, \dots, t_n] \cong \text{Sym}_B N_{B/A} \cong B[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$.

Specialise to the case of one equation:

$I=(x)$ $(A)/(x)^2$ is free over $B=A/(x)$

Then: $\Gamma(dx) = \bigoplus_{v \geq 0} B dx^{(v)}$, $|dx^{(v)}| = 2v$



\leadsto 2-periodic eventually!

$$\tilde{\partial}^2 = x \cdot \frac{\partial}{\partial x}$$

A regular local, $B=A/(x)$ as before, M any f.g. B -module.
Note: A and B are Cohen-Macaulay

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$$\dim A = N \quad (\Rightarrow) \quad \exists x \neq 0 \quad \dim B = N-1$$

Depth lemma (depth of M = max length of a regular sequ. on M)

$$0 \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow 0$$

If $\text{depth}(M_2) > \text{depth}(M_1)$, then $\text{depth } M_3 = \text{depth } M_1 + 1$

Thus, for: $0 \leftarrow M \leftarrow F_0 \leftarrow \Omega_B M \leftarrow 0$

↑ free B -module, $\text{depth } F_0 = N-1$

$$\Rightarrow 0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \dots \leftarrow F_{N-2} \leftarrow \Omega_B^{N-1}(M) \leftarrow 0$$

$$\text{depth}_B \Omega_B^{N-1} M = N-1 = \dim B \text{ or } \Omega_B^{N-1} M = 0, \text{ i.e. } \text{proj. dim } M < \infty.$$

Koszul-Buchsbaum formula:

$$\underbrace{\text{proj dim}_A M}_{\text{depth}_B M} + \text{depth}_A M = \dim A$$

If we take $M' = \Omega_B^{N-1} M (\neq 0)$

$$\text{proj dim}_A M' + N-1 = N$$

$$\Rightarrow \text{proj dim}_A M' = 1.$$

Therefore M' has a proj. resolution over A of the form

$$0 \rightarrow G \xrightarrow{\psi} F \rightarrow M' \rightarrow 0$$

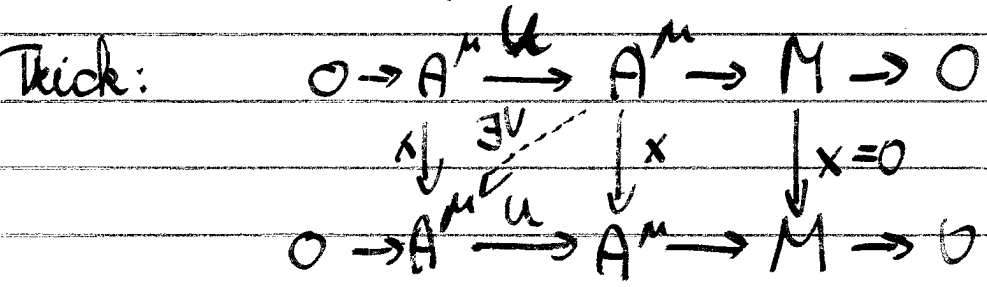
↑ f.g. free A -modules

$$M = M'$$

Note that M is A -torsion, as $x \cdot M = 0$, thus $\text{rank}_A F = \text{rank}_A G = \mu$

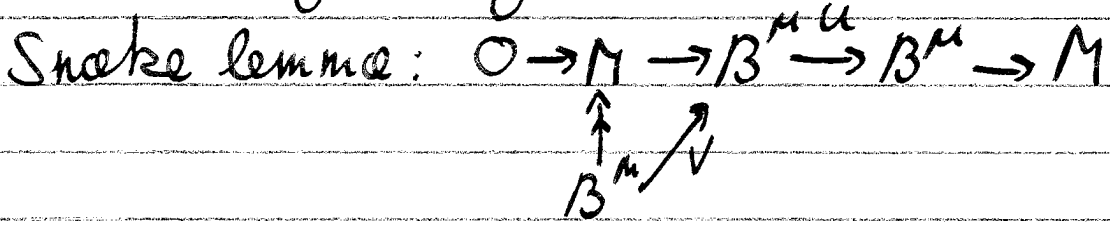
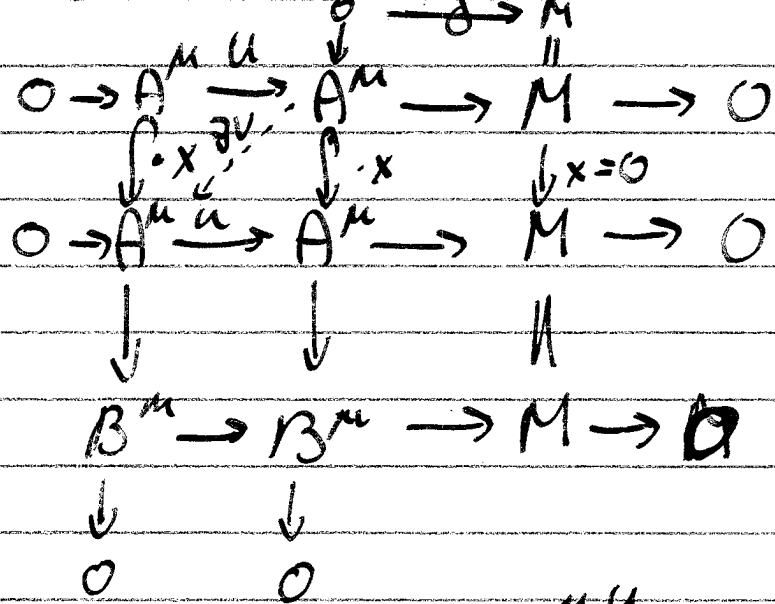
$$(*) \quad 0 \rightarrow A^\mu \xrightarrow{U} A^\mu \rightarrow M \rightarrow 0$$

where U is a $\mu \times \mu$ matrix over A .



U, V are $\mu \times \mu$ matrices over A , $VU = x \cdot \text{Id}_\mu \Rightarrow UV = x \cdot \text{Id}_\mu$
 (U, V) : a matrix factorization of x .

Complete the above diagram:



Thm Given a f.g. B -module M , it has a free B resolution

$$0 \leftarrow M \leftarrow F_0 \leftarrow \dots \leftarrow F_{n-2} \leftarrow F_{n-1} \xleftarrow{u} G \otimes_A B \xleftarrow{v} F \otimes_A B$$

$$\dots \xleftarrow{u} \dots \xleftarrow{v} \dots$$

(allow $F = G = 0$, i.e. $\text{projdim}_B M < \infty$)

$(U, V) \quad \text{MF}(x) \rightsquigarrow M = \text{coker } U \rightsquigarrow U$ up to base change from the minimal A -resolution of $\text{coker } U$.
 $\rightsquigarrow V$ from the diagram

Functor: $\text{MF}(x) \longrightarrow \text{MCM}(B)$
 = maximal-Cohen-Macaulay
 \Leftrightarrow depth of module is maximal

Minimality?

By choice, $\alpha = 0 \pmod{m_A}$

ex: $M = B$

$$0 \rightarrow A \xrightarrow{y} A \rightarrow B \rightarrow 0$$

$$\begin{matrix} & \times \downarrow & \nearrow & \downarrow \times & \downarrow \times = 0 \\ & & & & \end{matrix}$$

$$0 \rightarrow A \rightarrow A \rightarrow B \rightarrow 0$$

(x, y) is a matrix factorization, $\text{coker} = B$.

Fact: Minimal matrix factorizations, i.e. $U \equiv V \equiv 0 \pmod{m_A}$ correspond precisely to MCM without free summands.

MCM(B): the stable category

obj: are the MCM's

morph: $\underline{\text{Hom}}_B(M_1, M_2) := \text{Hom}_B(M_1, M_2) / \{ \varphi: M_1 \rightarrow \text{free} \rightarrow M_2 \}$

[in part: free mod become 0 in this category]

Thm $\underline{\text{MCM}}(B) \simeq \underline{\text{MF}}(x)$

MF(x): obj: matrix factorizations

mor: $G \xrightarrow{u} F \xrightarrow{v} G$

$\begin{array}{ccccc} \alpha \downarrow & & \downarrow & & \downarrow \beta \\ G & \rightarrow & F & \rightarrow & G \end{array}$

$\alpha, \beta \in \text{Hom}_A$
modulo homology

Resolutions over c.i.

A local, $B = A((x_1, \dots, x_n))$:

$I = (x_1, \dots, x_n)$

$I/I^2 \cong \bigoplus_{i=1}^n B(x_i \text{ mod } I)$ is free

M a (f.g.) B -module

$0 \leftarrow M \leftarrow \tilde{F}_0 \leftarrow \tilde{F}_1 \leftarrow \dots \leftarrow \tilde{F}_r \leftarrow \dots$ free A -resolution

$P^B(I/I^2[2]) = P^B(\tilde{F}) = P^B(dx_1, \dots, dx_n) \quad |dx_i| = 2$

$P^A(dx_1, \dots, dx_n)$

Structure theorem for resolutions:

On $\mathcal{F} = \tilde{\mathcal{F}} \otimes \mathcal{T}^A(dx_1, \dots, dx_n)$ there exists

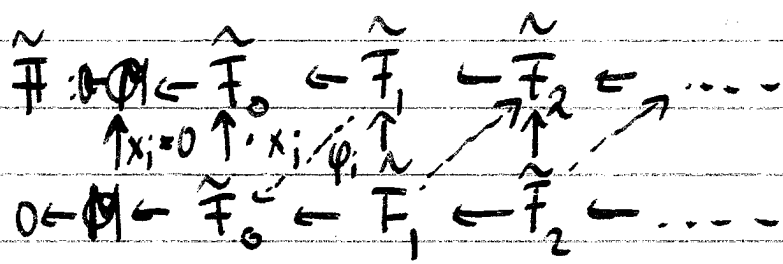
$\nabla : \mathcal{F} \rightarrow \mathcal{F}[1]$

$\nabla = \sum_{I \in \mathbb{N}^n} \varphi_I \frac{\partial^{|I|}}{\partial x^I}$ s.t. $\varphi_0, \dots, \varphi_n = \tilde{\partial}$
 $\varphi_I : \mathcal{F} \rightarrow \tilde{\mathcal{F}}[1 - 2|I|]$

$\nabla^2 = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$

Moreover, $(\mathcal{F} \otimes_A B, \nabla \otimes_A B)$ resolves M over B .

[Question in paper: can one avoid that the φ_I are power series?
 → No (see above)]



like in hypersurface

$\exists \varphi_i : \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}[-i]$
 $[\tilde{\partial}, \varphi_i] = x_i$, i.e. $\tilde{\partial} \varphi_i + \varphi_i \tilde{\partial} = x_i$
 → graded commutator

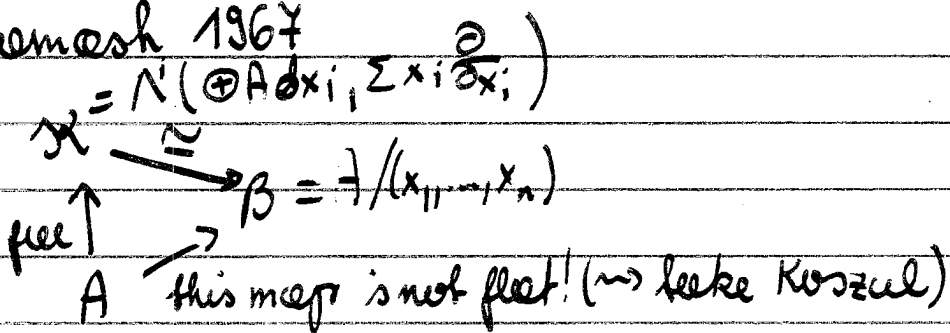
Do this for all i :

$\tilde{\mathcal{F}} \otimes \tilde{\partial}$
 $\uparrow \sum y_i \frac{\partial}{\partial x_i}$
 $\tilde{\mathcal{F}} \otimes \mathcal{T}(dx_1, \dots, dx_n)[2]$

Here: $\varphi_i \varphi_j + \varphi_j \varphi_i \sim 0$
 $\varphi_i^2 \varphi_j \sim 0$
 This gives
 deg 2 component of ∇ .

Go on... in paper: show that this process is finite (with spectral sequence)

cf. Sheemesh 1967



Want to resolve M as a \mathcal{K} -module:

$$0 \leftarrow M \leftarrow \tilde{\mathcal{F}}_0 \xrightarrow{\partial} \tilde{\mathcal{F}}_1 \xrightarrow{\partial} \dots \xrightarrow{\partial}$$

is a DG \mathcal{K} -module!

If you choose $\tilde{\mathcal{F}}$ such that $\tilde{\mathcal{F}}$ is a DG \mathcal{K} module (still free as $\alpha.$ of A , modules) then $\nabla = \partial + \sum_{i=1}^n \varphi_i \frac{\partial}{\partial x_i}$, where φ_i describes the action of dx_i on $\tilde{\mathcal{F}}$.

Fact: $D^{\bullet}(\text{DG-}\mathcal{K}\text{-modules}) \cong D^{\bullet}(B)$

Take the exterior algebra on the dx_i 's: Λ^{\bullet}
 $\uparrow \downarrow$ proj in deg 0
 A
 (graded)

11. Certain: the minimal resolution of the augmentation module A as Λ^{\bullet} -module is of the form, $\Lambda^{\bullet} \cong \mathbb{F} \langle dx_1, \dots, dx_n \rangle \oplus T^{\bullet}(dy_1, \dots, dy_n) \otimes_{\mathbb{F}} \Lambda^{\bullet}(\sum dx_i \frac{\partial}{\partial y_i})$ $|dy_i| = L$ $|dx_i| = 1$

$$\Rightarrow \text{Tor}_A^1(A, A) = 0 \quad [\text{cf. Illusie-Quillen}]$$

$$\begin{array}{ccc} \Lambda^{\bullet} \otimes \Lambda^{\bullet} & \cong & \Lambda^{\bullet}(\underline{dx}', \underline{dx}'') \\ \mu \downarrow & & \downarrow \tau \\ \Lambda^{\bullet} & = & \Lambda^{\bullet}(\underline{dx}) \end{array}$$

$$\Lambda^{\bullet}(\underline{dx}') \otimes T(\underline{dy}) \otimes \Lambda^{\bullet}(\underline{dx}') \quad \partial = \sum (dx'_1, \dots, dx'_n) \frac{\partial}{\partial y_i}$$

is a resolution of μ

If $\tilde{\#}$ is a DG K -resolution of M , then $(\tilde{\#} \otimes_{\Lambda} (\Lambda^{\bullet} \otimes_A T^{\bullet} \otimes_A \Lambda) \otimes_{\Lambda} B, \partial)$ is a B -resolution of M .

