Matrix factorization seminar 19/02/2013

Ragnar Buchweitz: About Eisenbud's paper

"Homological algebra on a complete intersection with an application to group representations"
TAMS 260, no. 1, (1980) 35-64.
submitted Febr. 1978

Two rings: A, B $A \to B = A/I$; A commutative
$I = (x_1, \dots, x_n)$, I/I^2 is free on the classes of the x_i as a basis.

Let M be a B-module and consider a free B-resolution
$0 \leftarrow M \leftarrow F_0 \xleftarrow{\partial} F_1 \xleftarrow{\partial} \dots \xleftarrow{\partial} F_k \leftarrow \cdots \quad : \mathbb{F}$

Note: Can represent the differentials as matrices over B.
Lift each F_i to a free A-module $\tilde{F}_i$ and each ∂ to a morphism $\tilde{\partial}$ of these free A-modules:
$\tilde{\mathbb{F}} = 0 \leftarrow \tilde{F}_0 \xleftarrow{\tilde{\partial}} \tilde{F}_1 \xleftarrow{\tilde{\partial}} \dots \xleftarrow{\tilde{\partial}} \tilde{F}_k \leftarrow \dots$

In general $\tilde{\partial} \neq 0$! But $\tilde{\mathbb{F}} \otimes_A B = \mathbb{F}$, $\tilde{\partial}^2 \otimes_A B = 0$.
$\Rightarrow \tilde{\partial}^2 = \sum_{i=1}^{n} x_i \tilde{t}_i$, $\tilde{t}_i : \tilde{\mathbb{F}} \to \tilde{\mathbb{F}}[2]$

Important fact: the $t_i := \tilde{t}_i \otimes_A B : \mathbb{F} \to \mathbb{F}[2]$
- are actual morphisms of cx.
- are unique up to B-homotopy
- are functorial up to homotopy

• the t_i, t_j commute up to homotopy.
[This is shown in the first section of the paper]

It is not relevant that $\mathcal{F}$ was a resolution. It suffices that $\mathcal{F}$ is a complex of B-free modules.

These facts can be summarized.

$$\tilde{D}(B) \simeq \mathbb{K} = \mathbb{K}^-(\underset{f.g.}{\mathrm{Free}}(B)) \qquad \begin{array}{l} id_{\mathbb{K}} \\ \downarrow t_i \in Z^2_{gr}(D^-(B)) \\ [2] \end{array}$$

Put these together and define: $Z^i_{gr}(D^-(B)) = \{ z : id \to [i] \mid z \circ [1] = [(-1)^i z][1] \}$

$$Z^\bullet_{gr}(B) := \bigoplus_i Z^i_{gr}(D^-(B))$$

graded com'm. ring under composition

We have a natural homomorphism of graded comm. rings. [Note: small category]

$$\underset{|t_i| = 2}{B[t_1, \dots, t_n]} \to Z^\bullet_{gr}(B)$$

The invariant description:

Thm (Mehta?): $B[t_1, \dots, t_n] = \mathrm{Sym}_B(\underbrace{\mathrm{Hom}_B(I/I^2, B)}_{N_{B/A}})$

$N_{B/A}$: normal module of B w.r.t. A.

$t_1, \dots, t_n$: the dual basis to $x_1, \dots, x_n$ mod I^2.

ex: A regular local $I = (x_1, \dots, x_n) \subseteq m_A$: maximal ideal of A.
Levin - Vasconcelos : In this situation
I/I^2 is B-free $\Longleftrightarrow (x_1, \dots, x_n)$ is an A-sequence
($\Longleftrightarrow x_1$ is a NZD on A, x_{i+1} is a NZD on $A/(x_1, \dots, x_i)$
$i = 1, \dots, n-1$, and $(x_1, \dots, x_n)A \neq A$)

Study the b_i on a minimal resolution of B/m_B as B-module
($I \subseteq m_A^2$).
This construction was given by J. Tate (1958): ~~Take~~
Assume that $m_A = (y_1, \dots, y_N)$ s.t. the y_j form an A-regular sequence. Take the Koszul cx. on the y_j:

$$\mathcal{K} = K(y_1, \dots, y_N) A = \bigoplus_{\nu \geq 0} \Lambda^\nu \Big(\bigoplus_{i=1}^{N} A\,dy_i \Big)[\nu] \text{ and}$$

$$\partial = \sum_{i=1}^{N} y_i \frac{\partial}{\partial y_i} \quad \text{(Euler derivation)}$$

This is a minimal resolution of $k = A/m_A$ over A (Serre).
Now Tate: Take $G = \bigoplus_{i=1}^{n} A\,dx_i$ and form the divided power algebra on G:

$$\Gamma_i G := \mathrm{Sym}_i (G^*)^*, \text{ where } * = \mathrm{Hom}_A(-, A).$$

$$\Gamma G = \bigoplus_{i \geq 0} \Gamma_i G.$$

The point here is: $\Gamma_i G$ has a basis $dx_1^{(k_1)}, \dots, dx_m^{(k_m)}$, $\sum_j k_j = i$

$$x_i^{(k)} = \frac{x^k}{k!} \qquad \frac{\partial x^{(k)}}{\partial x} = x^{(k-1)}$$

Consider: $\tilde{\mathcal{F}} = \Lambda^\bullet \Big(\bigoplus_{j=1}^{N} A\,dy_j [1] \Big) \otimes_A \Gamma^\bullet \Big(\bigoplus_{i=1}^{n} A\,dx_i [2] \Big)$

$$\tilde{\partial} = \sum_{j=1}^{N} y_j \frac{\partial}{\partial y_j} + \sum_{i=1}^{n} \sum_{j=1}^{N} x'_{ij} \frac{\partial}{\partial x_i} \quad \text{such that}$$

$$\sum_j x'_{ij} y_j = x_i \quad (x'\text{s unique modulo } I)$$

<u>Facts</u>: $\tilde{\partial}^2 = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \implies t_i$ correspond to $\frac{\partial}{\partial x_i}$.

Tate: $F = (\tilde{F}, \tilde{\partial}) \otimes_A B$ is the (minimal resolution of B/m_B over B

$$\tilde{\partial} = \sum_{i=1}^{n} x_i \tilde{t}_i.$$

Now: $B[t_1, \dots, t_n] \cong \mathrm{Sym}_B N_{B/A} \cong B[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$.

Specialise to the case of one equation:

$I = (x) \qquad (x)/(x)^2$ is free over $B = A/(x)$

Then: $\Gamma(dx) = \bigoplus_{\nu \geq 0} B\, dx^{(\nu)}, \quad |dx^{(\nu)}| = 2\nu$

deg 0

$$\begin{tikzcd} K \arrow[loop right, "\sum y_j \frac{\partial}{\partial y_j}"] & \\ & K \otimes dx \arrow[ul] \arrow[loop right, "\sum x' \frac{\partial}{\partial x}"] & \\ & & K \otimes dx^{(2)} \arrow[ul] \arrow[loop right] \end{tikzcd}$$

$\rightsquigarrow$ 2-periodic eventually!

$$\boxed{\tilde{\partial}^2 = x \cdot \frac{\partial}{\partial x}}$$

A regular local, $B = A/(x)$ as before, M any f.g. B-module.

<u>Note</u>: A and B are Cohen-Macaulay

$$\dim A = N \overset{\exists x \neq 0}{(\Rightarrow)} \dim B = N-1$$

Depth lemma (depth of M = max length of a regular sequ. on M)

$$0 \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow 0$$

If $\operatorname{depth}(M_2) > \operatorname{depth}(M_1)$, then $\operatorname{depth} M_3 = \operatorname{depth} M_1 + 1$

Thus, for: $0 \leftarrow M \leftarrow F_0 \leftarrow \Omega M \leftarrow 0$ ← syzygy module

F_0: free B-module, $\operatorname{depth} F_0 = N-1$

$$\Rightarrow 0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \dots \leftarrow F_{N-2} \leftarrow \Omega_B^{N-1}(M) \leftarrow 0$$

$$\operatorname{depth}_B \Omega_B^{N-1} M = N-1 = \dim B \text{ or}$$
$$\Omega_B^{N-1} M = 0, \text{ i.e. proj.dim } M < \infty.$$

Auslander-Buchsbaum formula:

$$\underbrace{\operatorname{projdim}_A M}_{} + \underset{\substack{\| \\ \operatorname{depth}_B M}}{\operatorname{depth}_A M} = \dim A$$

If we take $M' = \Omega_B^{N-1} M \; (\neq 0)$

$$\operatorname{projdim}_A M' + N-1 = N$$
$$\Rightarrow \operatorname{projdim}_A M' = 1.$$

Therefore M' has a proj. resolution over A of the form

$$0 \to G \xrightarrow{\varphi} F \to M' \to 0$$

G, F: f.g. free A-modules

$M = M'$

Note that M is ~~A-~~ torsion, as $x \cdot M = 0$, thus $\operatorname{rank}_A F = \operatorname{rank}_A G =: \mu$

$$(*) \quad 0 \to A^\mu \xrightarrow{\alpha} A^\mu \to M \to 0$$

where α is a $\mu \times \mu$ matrix over A.

Trick:

$$\begin{tikzcd}
0 \arrow[r] & A^\mu \arrow[r, "U"] \arrow[d, "x"'] & A^\mu \arrow[r] \arrow[d, "x"] \arrow[dl, dashed, "\exists V"'] & M \arrow[r] \arrow[d, "x=0"] & 0 \\
0 \arrow[r] & A^\mu \arrow[r, "U"] & A^\mu \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

U, V are $\mu \times \mu$ matrices over A, $VU = x \cdot \mathrm{Id}_\mu \Rightarrow UV = x \cdot \mathrm{Id}_\mu$

(U, V): a <u>matrix factorization</u> of x.

Complete the above diagram:

$$\begin{tikzcd}
 & & & M \arrow[d, equal] & \\
0 \arrow[r] & A^\mu \arrow[r, "U"] \arrow[d, "\cdot x"'] & A^\mu \arrow[r] \arrow[d, "\cdot x"] \arrow[dl, dashed, "\exists V"'] & M \arrow[r] \arrow[d, "x=0"] & 0 \\
0 \arrow[r] & A^\mu \arrow[r, "U"] \arrow[d] & A^\mu \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
 & B^\mu \arrow[r] \arrow[d] & B^\mu \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Snake lemma:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & B^\mu \arrow[r, "U"] & B^\mu \arrow[r] & M \\
 & B^\mu \arrow[u] \arrow[ur, "V"'] & & &
\end{tikzcd}$$

<u>Thm</u> Given a f.g. B-module M, it has a free B-resolution

$$0 \leftarrow M \leftarrow F_0 \leftarrow \dots \leftarrow F_{N-2} \leftarrow F\otimes_A B \xleftarrow{U} G\otimes_A B \xleftarrow{V} F\otimes_A B \;\cdots \xleftarrow{U} \cdots \xleftarrow{V} \cdots$$

(allow $F = G = 0$, i.e. $\operatorname{projdim}_B M < \infty$)

(U,V) $\quad MF(x) \rightsquigarrow M = \operatorname{coker} U \rightsquigarrow U$ up to base change from the minimal A-resolution of $\operatorname{coker} U$.
$\rightsquigarrow V$ from the diagram

Functor: $\quad MF(x) \longrightarrow \underline{MCM}(B)$
= maximal-Cohen-Macaulay
$\Leftarrow$ depth of module is maximal

Minimality?
By choice, $U = 0 \bmod m_A$

<u>ex</u>: $M = B$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\cdot y"] \arrow[d, "x"'] & A \arrow[r] \arrow[d, "\cdot x"] \arrow[dl, "1"'] & B \arrow[r] \arrow[d, "x=0"] & 0 \\
0 \arrow[r] & A \arrow[r] & A \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

$(x,1)$ is a matrix factorization, $\operatorname{coker} = B$.

<u>Fact</u>: <u>Minimal</u> matrix factorizations, i.e. $U \equiv V \equiv 0 \bmod m_A$ correspond precisely to MCM without free summands.

$\underline{MCM}(B)$: the stable category
obj: are the MCM's
morph: $\underline{Hom}_B(M_1, M_2) := Hom_B(M_1, M_2) / \{\varphi : M_1 \to \text{free} \to M_2\}$

[in part: free mod become 0 in this category]

<u>Thm</u> $\underline{MCM}(B) \simeq \underline{MF}(x)$

$\underline{MF}(x)$: obj: matrix factorizations
mor:

$$\begin{tikzcd} G \arrow[r, "u"] \arrow[d, "\beta"] & F \arrow[r, "v"] \arrow[d, "\alpha"] \arrow[dl, dashed, "\delta"'] & G \arrow[d, "\beta"] \arrow[dl, dashed, "\gamma"'] \\ G \arrow[r] & F \arrow[r] & G \end{tikzcd}$$

$\alpha, \beta \in Hom_A$
modulo homology

<u>Resolutions over c.i.</u>

A local, $B = A/(x_1, \dots, x_n)$: $I = (x_1, \dots, x_n)$
$I/I^2 \cong \bigoplus_{i=1}^{n} B\,(x_i \bmod I)$ is free

M a (f.g.) B-module

$$0 \leftarrow M \leftarrow \underbrace{\tilde{F}_0 \xleftarrow{\tilde{\partial}} \tilde{F}_1 \leftarrow \dots \xleftarrow{\tilde{\partial}} \tilde{F}_k \leftarrow \dots}_{\tilde{F}}$$ free A-resolution

$$P^B_\bullet(I/I^2[2]) = P^B_\bullet(dx_1, \dots, dx_n) \qquad |dx_i| = 2$$

$$P^A_\bullet(dx_1, \dots, dx_n)$$

Structure theorem for resolutions:

On $\mathbb{F} = \tilde{\mathbb{F}} \otimes_A \Gamma^A(dx_1, \dots, dx_n)$ there exists

- $\nabla : \mathbb{F} \to \mathbb{F}[1]$
- $\nabla = \sum_{I \in \mathbb{N}^n} \varphi_I \frac{\partial^I}{\partial \underline{x}^I}$ s.t. $\varphi_{0,\dots,0} = \tilde{\partial}$
  $\varphi_I : \tilde{\mathbb{F}} \to \tilde{\mathbb{F}}[1 - 2|I|]$
- $\nabla^2 = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$

Moreover, $(\mathbb{F} \otimes_A B, \nabla \otimes_A B)$ resolves M over B.

[Question in paper: can one avoid that the φ_I are power series?
→ No (see Avramov)]

$$\begin{tikzcd}
\tilde{\mathbb{F}} : 0 & M \arrow[l] & \tilde{F}_0 \arrow[l] & \tilde{F}_1 \arrow[l] & \tilde{F}_2 \arrow[l] & \cdots \arrow[l] \\
0 & M \arrow[l] \arrow[u, "x_i = 0"] & \tilde{F}_0 \arrow[l] \arrow[u, "x_i"] & \tilde{F}_1 \arrow[l] \arrow[u] \arrow[ul, dashed, "\varphi_i"] \arrow[ur, dashed] & \tilde{F}_2 \arrow[l] \arrow[u] \arrow[ur, dashed] & \cdots \arrow[l]
\end{tikzcd}$$

like in hypersurface
$\exists\, \varphi_i : \tilde{\mathbb{F}} \to \tilde{\mathbb{F}}[-1]$
$[\tilde{\partial}, \varphi_i] = x_i$, i.e. $\tilde{\partial}\varphi_i + \varphi_i \tilde{\partial} = x_i$
graded commutator

Do this for all i:

$$\begin{tikzcd}
\tilde{\mathbb{F}} \arrow[loop right, "\tilde{\partial}"] \\
\tilde{\mathbb{F}} \otimes \Gamma(dx_1, \dots, dx_n)[2] \arrow[u, "\sum \varphi_i \frac{\partial}{\partial x_i}"] \arrow[loop right]
\end{tikzcd}$$

Here: $\varphi_i \varphi_j + \varphi_j \varphi_i \sim 0$
$\varphi_i^2 \sim 0$
This gives
deg 2 component of ∇.

Go on... in paper: show that this process is finite
(with spectral sequence)

cf. Shamash 1967

$$\begin{tikzcd} K = \Lambda^\cdot(\oplus A\,dx_i, \sum x_i \frac{\partial}{\partial x_i}) \arrow[r, "\simeq"] & B = A/(x_1,\dots,x_n) \\ A \arrow[u, "\text{free}"] \arrow[ur] & \end{tikzcd}$$

this map is not flat! ($\rightsquigarrow$ take Koszul)

Want to resolve M as a K module:

$$0 \leftarrow M \leftarrow \underbrace{\tilde{F}_0 \xleftarrow{\partial} \tilde{F}_1 \xleftarrow{\partial} \dots \xleftarrow{\partial}}_{\text{is a DG } K\text{-module!}}$$

If you choose $\tilde{F}$ such that $\tilde{F}$ is a DG K module (still free as c. of A-modules) then $\nabla = \partial + \sum_{i=1}^{n} \varphi_i \frac{\partial}{\partial x_i}$, where φ_i describes the action of dx_i on $\tilde{F}$.

Fact: $D^\cdot(\text{DG-}K\text{-modules}) \simeq D^\cdot(B)$

Take the exterior algebra on the dx_i's: $\Lambda^\cdot$

$$\begin{tikzcd} \Lambda^\cdot \arrow[d, shift left, "\text{proj in deg } 0"] \\ A \arrow[u, shift left] \end{tikzcd}$$

H. Cartan: the minimal (graded) resolution of the augmentation module A as $\Lambda^\cdot$-module is of the form ($\Lambda^\cdot = A\langle dx_1,\dots,dx_n\rangle$)

$(\Gamma^\cdot(dy_1,\dots,dy_n)\otimes_A \Lambda^\cdot, \sum dx_i \frac{\partial}{\partial y_i})$ $\quad |dy_i| = 2$, $|dx_i| = 1$

$\Rightarrow \mathrm{Tor}_i^{\Lambda^{\bullet}}(A, A) = \Gamma$ [cf. Illusie-Quillen]

$$\begin{array}{ccc} \Lambda^{\bullet} \mathcal{P} \otimes_A \Lambda^{\bullet} & \cong & \Lambda^{\bullet}(\underline{dx}', \underline{dx}'') \\ \mu \downarrow & & \downarrow \begin{smallmatrix} \underline{dx}', \underline{dx}'' \\ \downarrow \\ \underline{dx} \end{smallmatrix} \\ \Lambda^{\bullet} & = & \Lambda^{\bullet}(\underline{dx}) \end{array}$$

$(\Lambda^{\bullet}(\underline{dx}') \otimes \Gamma(\underline{dy}) \otimes \Lambda^{\bullet}(\underline{dx}'))$ $\quad \partial = \sum (dx'_1, \ldots, dx'_n) \frac{\partial'}{\partial y_i}$

is a resolution of μ

If $\tilde{F}$ is a DG K-resolution of M, then $(\tilde{F} \otimes_{\Lambda^{\bullet}} (\Lambda^{\bullet} \otimes_A \Gamma' \otimes_A \Lambda) \cdots$

$\cdots \otimes_{\Lambda} B, \partial)$

is a $\hat{B}$-resolution of M.