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Ragnar Buchweitz: Hochster's Theta Pairing + m.f.

Recall: If $R = S/(f)$, S regular and M any (f.g.) R-module then it has an eventually 2-periodic proj resolution

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \dots \leftarrow P_i \xleftarrow{A} P_{i+1} \xleftarrow{B} P_i \xleftarrow{A} P_{i+1} \leftarrow \dots$$

In part. $\mathrm{Tor}_i^R(M,N)$ only depends on $i \bmod 2$ for $i \gg 0$.
If $\mathrm{Tor}_i^R(M,N)$ has finite length eventually, then

$$\Theta(M,N) := \lg(\mathrm{Tor}^R_{even}(M,N)) - \lg \mathrm{Tor}_{odd}(M,N))$$

(where $\mathrm{Tor}_{even} := \mathrm{Tor}_{2i}$, $\mathrm{Tor}_{odd} := \mathrm{Tor}_{2i+1}$)

Up to sign, $\Theta(M,N) = \chi(N, M^*)$ if M is MCM

$$\chi(N,M^*) = \lg \mathrm{Ext}^{even}_R(M^*,N) - \lg \mathrm{Ext}^{odd}_R(M^*,N)$$

"Herbrand difference"

$\rightsquigarrow$ Θ or χ can be viewed as the Euler characteristic on $D^{st}_{sg}(R) \cong \underline{MCM}(R)$

Hochster proves that "all" homological conjectures on comm. rings will follow if one can prove $\Theta(R/\mathfrak{p}, -) = 0$ for a specific hypersurface R and prime ideal $\mathfrak{p}$ (still open!)

$$R = \underbrace{\Delta(x_1,\dots,x_n; y_1,\dots,y_n; z_1,\dots,z_n)_{\mathfrak{m}}}_{S} / F_k$$

Δ is a DVR, $\mathfrak{p}$ the maximal ideal, $\mathfrak{m}$: max ideal containing $\mathfrak{p}, \underline{x}, \underline{y}, \underline{z}$, $F_k = x_1^k \cdots x_n^k - \sum_{i=1}^n y_i x_i^{k+1}$

DSC $\Leftrightarrow$ $V = V((x_1,\dots,x_n)R)$, then $W \subseteq \operatorname{Spec} R$ Hochster
$V \cap W = \{\mathfrak{m}\} \Rightarrow \dim V + \dim W \leq \dim R \Leftrightarrow \Theta(R/(\underline{x}), -) = 0$

Serre's definition of intersection multiplicity:
$\chi(M,N) = \sum_i (-1)^i \lg \operatorname{Tor}_i^R(M,N)$ for R regular,
$\lg(M \otimes N) < \infty$.

Prop (Hochster): $M = R/I$, $N = R/J$ $R/I+J$ artinian,
then $\Theta(M,N) = \chi_S(M,N)$

Thm (Moore - Piepmeyer - Spiroff - Walker: graded, cont. a field
Polishchuk - Vaintrob: formal, char 0,
Buchw. - van Straten: analytic case; char 0)
If $R = S/(f)$ a local isolated hypersurface sing.,
$\dim R$ even, then $\Theta(-,-) \equiv 0$. (conj. by H. Dao 2007)

Properties of Θ (a) Θ defines a biadditive pairing
$K_0(R) \times K_0(R) \to \mathbb{Z}$
(b) Θ descends to $K_0(R)/K_0(\text{proj. } R)$
(c) Θ vanishes on torsion elements, since $\mathbb{Z}$ is torsionfree.
e: If R is an evendim. single sing. then $\Theta \equiv 0$
because $K_0(R)/[R]$ is torsion.
(d) Θ kills artinian modules.

Consider the resolution of $k = R/\mathfrak{m}$, $\mathfrak{m}$ a max ideal
$0 \leftarrow M \leftarrow \dots \leftarrow P_i \xleftarrow{A} P_{i+1} \xleftarrow{B} P_i \xleftarrow{A}$ $\operatorname{rank} P_i = \operatorname{rk} P_{i+1}$
$\operatorname{Tor}_i^R(M,k) \equiv \dim_k P_i \otimes k = \dim_k P_{i+1} \otimes k \Rightarrow \Theta(M,k) = 0$. $\square$

So Θ descends to $K_0(R)/K_0(\mathrm{proj}\,R) + \text{Artinian})$.

Note: $\mathrm{mod}\, R/\mathrm{art}\, R \cong \mathrm{Coh}\, U \qquad U = \mathrm{Spec} R \setminus \{m\}$
this group depends on U
$$\Theta(M,N) = \Theta(M|_U, N|_U)$$

(e) Θ vanishes on divisible elements

Motto: Θ is "extremely degenerate".

Indications of proof of Thm:
Use an appropriate version of Hirzebruch-Riemann-Roch for suitable cohomology theory:
MPSW: H_{et}
P-B: HH_{MF} à la Dyckerhoff
B-vS: $H(\mathrm{link})$

f homogeneous in $K[x_1, \dots, x_{2m+2}]$ $\deg f = d$.
$X = V(f=0) \subseteq \mathbb{P}^{2m+1}$, $Y, Z \subseteq X$ cycles of dim m that intersect transversally.

Prop (B-vS) Set $M = S/I(Y)$, $N = S/I(Z)$, then
$$\Theta(M,N) = -1/d\, [|Y|] \cdot [|Z|], \text{ where}$$

$[|Y|] = [Y] - (\deg Y) h^m \in H^{2m}(X)$; h hyperplane section
$[Y]$ fundamental class of Y.

equivalently, $\Theta(X,Y)$ is the intersection number in the primitive cohomology of X.

$X \subseteq \mathbb{P}^{2m+1}$: $H^\bullet(\mathbb{P}) \xrightarrow[\substack{\text{isom except}\\ \text{and } \bullet = m}]{} H^\bullet(X) \to$ primitive cohom $\to 0$

$\dim X = 2m$

Hodge decomposition $H^k(X) \cong \bigoplus_{p+q=k} H^p(X, \Omega_X^q)$

$H(\mathbb{P}) = \bigoplus H^i(X, \Omega_X^i)$;

P. Griffiths: $H^{q}(X, \Omega_X^p) = k[x_1, \dots, x_{2n+1}] / \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{2n+2}}\right)$

$p+q = m$ in degree $d(p+1) - 2m - 2$

f = smooth cubic surface in $\mathbb{P}^3$, $m=1$, $d=3$

$\mathrm{Jac}(f) = k[x_1, \dots, x_4] / (\text{c.i. of 4 quadrics})$

Hilbert fct:

0	4
1	6
:	4
4	1

E_6-lattice in $H^1(f=0)$ (with its natural intersection form) even without transversality!

Proof of $\Theta(Y, Z) = -\frac{1}{d}[|Y|][|Z|] = -d[Y][Z] + \deg Y \cdot \deg Z$

$M = S/I(Y)$, $N = S/I(Z)$

$\mathbb{H}(M \otimes_R^{\mathbb{L}} N) = \dfrac{\mathbb{H}(M) \cdot \mathbb{H}(N)}{\mathbb{H}(R)}$, where $\mathbb{H} = \sum_{i \in \mathbb{Z}} \dim_k M_i t^i$. (*)

$\mathbb{H}(M \otimes_R^{\mathbb{L}} N) = \mathbb{H}(M \otimes_R N) + \text{polynomial} + \dfrac{\mathbb{H}(\mathrm{Tor}^R_{\mathrm{even}}(M,N)) - \mathbb{H}(\mathrm{Tor}^R_{\mathrm{odd}}(M,N))}{1 - t^d}$

If M, N intersect transversally then this is well-defined $\mathbb{H}(M \otimes_R^{\mathbb{L}} N) = \underbrace{P(M) \cdot P(N)}_{\text{polynom.}} / (1-f)^{n+1} \rightsquigarrow \operatorname{res}_1(*) = \frac{1}{d}\,\Theta(\mathcal{O}_Y, \mathcal{O}_Z)$ $\square$

Supplement to Thm (MPSW): Θ

If $\dim R = n$ odd, then $(-1)^{\frac{n+1}{2}} \cdot$ isolated sing

is positive definite if char $k = 0$.

Ref: arXiv, Moscow J.

From now on, $R = \mathbb{C}\{\underline{x}\}/(f)$ isolated analyt. singularity

Thm (B.-van Straten) Let $f \in P = \mathbb{C}\{x_1, \dots, x_n\}$ define an isolated singularity, let M, N be $R = P/(f)$-modules

(i) If n is odd, then $\Theta(M,N) = 0$

(ii) If n is even, then $\Theta(M,N) = lk(\operatorname{ch} M, \operatorname{ch} N)$
where $\operatorname{ch} : K^0(\text{link}) \to H^{ev}(L)$ $\quad L$ = link of the singularity

In fact, if $n = 2m$, then only $\operatorname{ch}^m(M)$, $\operatorname{ch}^m(N)$ contribute.
$\Theta(M,N) = lk(\operatorname{ch}^m M, \operatorname{ch}^m N)$.

Idea of the proof:

(1) Interpretation in algebraic K-theory

A: abelian (exact) category, K-theory à la Quillen

$K_0(A)$: Grothendieck group of A.

Localization sequence: $S \subseteq A$ Serre subcategory

long exact sequence:

$$\sim \to K_i(S) \to K_i(A) \to K_i(A/S) \xrightarrow{\partial} K_{i-1}(S) \to \dots$$

Nenashev; Sherman show that elements in $K_1(A)$ can be represented as "double short exact sequences":

$$\Xi \equiv \begin{pmatrix} 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \\ \\ 0 \to A \xrightarrow{h} B \xrightarrow{k} C \to 0 \end{pmatrix}$$

(not commutative!) [Ref: Gillet, Grayson]

typical case:

$$\begin{bmatrix} 0 \to 0 \to B = B \to 0 \\ \\ 0 \to 0 \to B \xrightarrow{\alpha} B \to 0 \end{bmatrix}$$

$\Updownarrow$

α is an automorph. of B

A cyclic diagram

$$\Sigma = \left[\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] & X \arrow[r, "\beta"] \arrow[d, equal] & C \arrow[r] \arrow[d, "\cong\ \nu"] & 0 \\ 0 & D \arrow[l] \arrow[u, "\mu\ \cong"] & Y \arrow[l, "\gamma"'] & B \arrow[l, "\delta"'] & 0 \arrow[l] \end{tikzcd}\right] \rightsquigarrow$$

with both rows s.e.s.

Rewrite :

$$\begin{tikzcd} 0 \arrow[r] & A \oplus B \arrow[r, "s"] & X \oplus B \oplus D \arrow[r, "t"] \arrow[d, equal] & C \oplus D \arrow[r] \arrow[d, "\cong"', "n = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}"] & 0 \\ 0 \arrow[r] & D \oplus B \arrow[r, "r"] \arrow[u, "\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} = m\ \cong"] & X \oplus B \oplus D \arrow[r, "s"] & B \oplus D \arrow[r] & 0 \end{tikzcd}$$

$$s = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 & 0 \\ \delta & 0 & 1 \end{pmatrix}$$

$$\leadsto \quad \Xi \equiv \begin{bmatrix} 0 \to A\oplus B \xrightarrow{p} X\oplus B\oplus D \xrightarrow{\varphi' := m\circ\varphi} B\oplus D \to 0 \\ \\ 0 \to A\oplus B \xrightarrow[\substack{\shortparallel \\ r\, m^{-1}}]{r'} X\oplus B\oplus D \xrightarrow{\quad} B\oplus D \to 0 \end{bmatrix} \in K_1(\mathcal{A})$$

A 2-periodic complex $C^\cdot$ in $\mathcal{A}$:

$$\dots \xrightarrow{a} Y \xrightarrow{b} X \xrightarrow{a} Y \to \dots$$

(Y in in even degrees, $Y = X$)

$$\begin{array}{ccccccccc} \xi \equiv 0 & \to & A & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & C & \to & 0 \\ & & \uparrow{\scriptstyle\mu} & & \shortparallel & & \downarrow{\scriptstyle\nu} & & \\ 0 & \leftarrow & D & \leftarrow & Y & \leftarrow & B & \leftarrow & 0 \end{array}$$

$A = \ker a$
$C = \operatorname{Im} a$
$B = \operatorname{Ker} b$
$D = \operatorname{Im} b$

(Sherman)
The boundary map $K_1(\mathcal{A}/S) \to K_0(S)$ $S \leq \underline{A}$ Serre in $\underline{A}$

If $f: A \to B$ is an S-isomorphism, then $\bar f : \bar A \xrightarrow{\sim} \bar B$ is the corresponding isom in S
$\chi(f) := [\operatorname{coker} f] - [\ker f] \in K_0(S)$.

On the other hand if A, B are lifts of $\bar A, \bar B \in \underline{A}/S$
$\varnothing : \bar A \xrightarrow{\sim} \bar B$ is an isom, we can lift: $A \xleftarrow{a} \cdot \xleftarrow{f} D \xleftarrow{b} B$

so that a, b are S-isomorphisms, $\phi = \bar{b}^{-1} \bar{f} \bar{a}^{-1}$

If f itself is an S-isomorphism, one gets
$$\chi(\phi) := \chi(f) - \chi(a) - \chi(b) \in K_0(S).$$

Now take $x \in K_1(A/S)$:

$$x = \begin{bmatrix} 0 \to \bar{A} \xrightarrow{\lambda} \bar{B} \xrightarrow{\mu} \bar{C} \to 0 \\ \\ 0 \to \bar{A} \xrightarrow{\bar{\sigma}} \bar{B} \xrightarrow{\bar{\tau}} \bar{C} \to 0 \end{bmatrix}$$

one can "lift" back to A in such a way that

$$\begin{tikzcd}
0 \arrow[r] & \bar{A}_1 \arrow[r, "\bar{l}"] \arrow[d, "\alpha_1"', "\cong"] & \bar{B}_1 \arrow[r, "\bar{m}"] \arrow[d, "\cong"', "\beta_1"] & \bar{C}_1 \arrow[r] \arrow[d, "\cong"', "\gamma_1"] & 0 \\
0 \arrow[r] & \bar{A} \arrow[r, "\lambda"] & \bar{B} \arrow[r, "\mu"] & \bar{C} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \bar{A} \arrow[r, "\bar{\sigma}"] & \bar{B} \arrow[r, "\bar{\tau}"] & \bar{C} \arrow[r] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "\bar{s}"] \arrow[u, "\alpha_2", "\cong"'] & \bar{B}_2 \arrow[r, "\bar{t}"] \arrow[u, "\cong", "\beta_2"'] & \bar{C}_2 \arrow[r] \arrow[u, "\cong", "\gamma_2"'] & 0
\end{tikzcd}$$

"an extended diagram for x"

<u>Thm</u> (Sherman) The boundary map $K_1(A/S) \xrightarrow{\partial} K_0(S)$

$$\partial(x) = \chi(\alpha) - \chi(\beta) + \chi(\gamma) \quad \text{where } \alpha = \alpha_2^{-1}\alpha_1,\ \beta = \beta_2^{-1}\beta_1,\ \gamma = \gamma_2^{-1}\gamma_1$$

<u>Cor</u>: If m is a cyclic diagram

$$\xi \equiv \begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "\alpha"] & X \arrow[r, "\beta"] \arrow[d, equal] & C \arrow[r] \arrow[d, "\nu"] & 0 \\ 0 & D \arrow[l] \arrow[u, "\mu"] & Y \arrow[l, "\delta"'] & B \arrow[l, "\gamma"'] & 0 \arrow[l] \end{tikzcd}$$

μ and ν are S-isomorphisms, then the reduction in <u>A/S satisfies</u> $\partial(\xi) = (\underline{\mathrm{Coker}(\nu)} - \underline{\mathrm{Ker}(\nu)}) - (\mathrm{coker}(\mu) - \mathrm{ker}(\mu))$

If M is an MCM over R, $[M] = [M|_{U = \mathrm{Spec} R \setminus \{m\}}] \in K^0(U)$: vector b. on U

$\{M\} = \{C^\bullet(M)|_U\} \in K^1(U) \} = K^1(\mathrm{mod} R/\mathrm{art} R)$

2-per. res. of M

natural product: $K^1(U) \times K^1(U) \to K^1(U)$ coming from $\otimes$,
a trace map $\chi := l \circ \partial_e \, K^1(U) \to K^0(k) \cong \mathbb{Z}$

[Complete the argument: apply forgetful functor to top. category ⇝ things are easier to handle there]